Preliminary Examination in Analysis

June 2022

Instructions

- This is a three-hour examination which consists of two parts: Advanced Calculus and Real or Complex Analysis.
- You should work problems from the section on advanced calculus and from the section of the option that you have chosen.
- You are to work a total of five problems (four mandatory problems and one optional problem).
- You must work two mandatory problems from each part.
- Please indicate clearly on your test paper which optional problem is to be graded.
- Indicate clearly what theorems and definitions you are using.

Advanced Calculus, Mandatory Problems

1. Let f be a continuous real-valued function on $\mathbb{R}$ with

$$\lim_{x\to\infty} f(x) = 1 \text{ and } \lim_{x\to-\infty} f(x) = 0.$$

Show that f is uniformly continuous on $\mathbb{R}$.

2. Let $\{a_n\}$ and $\{b_n\}$ be two bounded sequences of real numbers. Recall that

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \left(\inf\{a_k | k \geq n\}\right).$$

a) Show that

$$\liminf_{n\to\infty}(a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n.$$

b) Show that if $\{b_n\}$ converges then

$$\liminf_{n\to\infty}(a_n + b_n) = \liminf_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

Advanced Calculus, Optional Problems

3. Consider the sequence defined recursively by

$$a_1 = 1, \quad a_{n+1} = 1 - \frac{1}{2 + a_n} = \frac{1 + a_n}{2 + a_n}.$$

Prove that this sequence converges, and determine its limit.

4. Let φ be a continuous function in $\mathbb{R}$ with compact support and let f be a bounded and uniformly continuous function in $\mathbb{R}$. Define

$$f_n(x) = n \int_{\mathbb{R}} f(x - y)\varphi(ny)\, dy.$$

Show that $f_n \to \alpha f$ uniformly in $\mathbb{R}$, where

$$\alpha = \int_{\mathbb{R}} \varphi(y)\, dy.$$

Real Analysis, Mandatory Problems

1. Let f, g be two Lebesgue integrable functions in $\mathbb{R}^d$. Assume that

$$\int_E f(x)\,dx \le \int_E g(x)\,dx$$

for any measurable subset E of $\mathbb{R}^d$. Show that $f \le g$ a.e. in $\mathbb{R}^d$.

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function whose derivative is bounded on $\mathbb{R}$, and $g : \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable. Show that

$$(f * g)(x) := \int_{\mathbb{R}} f(x-y)g(y)\,dy$$

is a differentiable function on $\mathbb{R}$, and

$$(f * g)'(x) = \int_{\mathbb{R}} f'(x-y)g(y)\,dy$$

Real Analysis, Optional Problems

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is in $L^1(\mathbb{R})$. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\sin(nx)\,dx = 0.$$

Hint: you may use the fact that step functions are dense in $L^1(\mathbb{R})$.

4. Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable set $E \subset \mathbb{R}^d$ with $m(E) < \infty$. Suppose that for each $x \in E$,

$$\sup\{|f_k(x)| : k \ge 1\} < \infty.$$

Show that for each $\epsilon > 0$ there exists a closed set F such that $m(E \setminus F) < \epsilon$ and

$$\sup\{|f_k(x)| : x \in F \text{ and } k \ge 1\} < \infty.$$

Complex Analysis, Mandatory Problems

1. Use the residue theorem to evaluate the integral

$$\int_0^\infty \frac{1}{\sqrt{x}(x^2+1)}\mathrm{d}x.$$

2. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that

$$|f(z)| \leq |z|^{10.5} \text{ for all } |z| \geq 100.$$

Prove that f is a polynomial and that its degree is at most 10.

Complex Analysis, Optional Problems

3. Let D be the unit disk $\{z \in \mathbb{C} : |z| < 1\}$. Find the number of solutions to the equation $e^z = 10z^{100}$ in D.

4. Suppose D is the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $f : D \to \mathbb{C}$ is a function with the property that f^k is analytic on D for all $k \in \mathbb{N}$ with $k > 1$. Show that f is also analytic on D.