Preliminary Examination in Analysis

June 2023

Instructions

- This is a three-hour examination which consists of two parts: Advanced Calculus and Real or Complex Analysis.
- You should work problems from the section on advanced calculus and from the section of the option that you have chosen.
- You are to work a total of five problems (four mandatory problems and one optional problem).
- You must work two mandatory problems from each part.
- Please indicate clearly on your test paper which optional problem is to be graded.
- Indicate clearly what theorems and definitions you are using.

Advanced Calculus, Mandatory Problems

1. Suppose A and B are bounded and nonempty subsets of $\mathbb{R}$ with the property that for every $b \in B$, there exists a sequence $\{a_n\}$ of elements of A which converges to b. Show that

$$\sup A \geq \sup B.$$

2. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Suppose further that f' is strictly increasing and that $f(a) = f(b) = 0$. Show that

$$f(x) < 0 \quad \text{for all } x \in (a, b).$$

Advanced Calculus, Optional Problems

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous with $f(0) = 0$ and that f is differentiable at 0. Show that there exists $C \in \mathbb{R}$ such that

$$|f(x)| \leq Cx \quad \text{for all } x \in [0, 1].$$

4. Suppose $a_n \geq 0$. Show that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges.

Real Analysis, Mandatory Problems

For a measurable subset E of $\mathbb{R}^d$, we use $m(E)$ to denote the Lebesgue measure of E.

1. Let E be a subset of $\mathbb{R}^d$. Define the interior measure of E by

$$|E|_i = \sup\big\{m(F):\ F \subset E \text{ and } F \text{ is closed}\big\}.$$

(a). Show that $|E|_i \le m_*(E)$, where $m_*(E)$ denotes the exterior measure of E.

(b). Suppose that $m_*(E) < \infty$. Show that E is measurable if and only if $|E|_i = m_*(E)$.

2. (a) Let $\{f_n\}$ be a sequence of measurable functions on a measurable set E with $m(E) < \infty$. Show that if $f_n(x) \to f(x)$ for a.e. $x \in E$, then for any $\alpha > 0$,

$$m\{x \in E : |f_n(x) - f(x)| > \alpha\} \to 0 \quad \text{as } n \to \infty;$$

i.e., $f_n \to f$ in measure on E.

(b) Show by an example that the conclusion in (a) may not be true without the assumption $m(E) < \infty$.

Real Analysis, Optional Problems

3. Let f be an integrable function in $\mathbb{R}^d$.

(a) Construct a sequence $\{f_n\}$ of bounded functions with compact support such that

$$\int_{\mathbb{R}^d} |f - f_n| \to 0 \quad \text{as } n \to \infty.$$

(b) Use part (a) to show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\int_E |f| < \varepsilon,$$

whenever E is measurable with $m(E) < \delta$.

4. (a) State the definition of an absolute continuous function f on $[a, b]$.

(b) State the definition of a function f of bounded variation on $[a, b]$.

(c) Use the definitions to show that if f is absolute continuous on $[a, b]$, then f is of bounded variation on $[a, b]$.

(d) Give an example of a function f that is continuous and of bounded variation on $[0, 1]$, but not absolutely continuous on $[0, 1]$. Prove your statement.

Complex Analysis, Mandatory Problems

1. Let f be analytic on the unit disk $\mathbb{D}$ and continuous on its closure $\overline{\mathbb{D}}$. Suppose that $|f(z)| = 1$ for all $|z| = 1$.

(a) Prove that if f never vanishes in $\mathbb{D}$, then f is a constant.

(b) Prove that there are only finitely many zeros of f in $\mathbb{D}$.

(c) Suppose $a_1, a_2, \ldots, a_n$ are the zeros of f in $\mathbb{D}$. Prove that there is an angle $\theta \in \mathbb{R}$ so that

$$f(z) = e^{i\theta} \left(\frac{z - a_1}{1 - \overline{a}_1 z} \right) \cdots \left(\frac{z - a_n}{1 - \overline{a}_n z} \right).$$

2. Compute the following integral

$$\int_0^\infty \frac{\cos(\pi x)}{(x^2+1)^2}\, dx.$$

Justify all steps of the calculation.

Complex Analysis, Optional Problems

3. Suppose that $f : \mathbb{D} \to \mathbb{C}$ and $\Re f(z) > 0$, for all $z \in \mathbb{D}$. Furthermore, assume $f(0) = 1$. Then, prove that f satisfies

$$\frac{1 - |z|}{1 + |z|} \leq |f(z)| \leq \frac{1 + |z|}{1 - |z|}$$

for all $z \in \mathbb{D}$. Hint: Construct a map $\mathbb{D} \to \mathbb{D}$ by composing f with an appropriate conformal map. Recall that the map $z \to \frac{i-z}{i+z}$ maps $\mathbb{H}$ biholomorphically onto $\mathbb{D}$.

4. Let $\{f_n\}$ be a sequence of analytic functions on a region $\mathcal{A} \subset \mathbb{C}$ converging uniformly on compact subsets of $\mathcal{A}$.

(1) Prove that a limit function f exists that is analytic on $\mathcal{A}$.

(2) Assume that f is not identically zero. Then, there exists a point $z_0 \in \mathcal{A}$ with $f(z_0) = 0$ if and only if there is a sequence $z_n \to z_0$ in $\mathcal{A}$ so that $f_n(z_n) \to 0$. HINT: Apply Rouché's Theorem.