Preliminary Examination in Analysis

June 2024

Instructions

- This is a three-hour exam on Advanced Calculus and Real or Complex Analysis.
- Please work a total of five problems (four mandatory problems, two from each section, and one optional problem). You *must* work the mandatory problems from each part
- Please indicate clearly on your test paper which optional problem is to be graded
- Please indicate clearly what theorems and definitions you are using

Advanced Calculus, Mandatory Problems

1. Suppose $\lim_{k\to\infty} a_k = L$. Show that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_k = L.$$

2. Suppose that f and g are Riemann integrable on $[a,b]$. Show that

$$h(x) = \max\{f(x), g(x)\}$$

is Riemann integrable on $[a,b]$.

Advanced Calculus, Optional Problems

3. Suppose that $f(x) \leq g(x) \leq h(x)$ for all $x \in \mathbb{R}$, and f and h are differentiable everywhere. Show that if $f(0) = g(0) = h(0)$ then g is differentiable at 0.

   Hint: First show that $f'(0) = h'(0)$.

4. Show that

$$\lim_{p\to\infty} \left(\int_0^{\pi} \sin^p(x)\, dx\right)^{\frac{1}{p}} = 1.$$

   Hint: You do not need to integrate $\sin^p(x)$ explicitly.

Real Analysis, Mandatory Problems

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is integrable. Show that, for any $\varepsilon > 0$, there is a bounded function g of compact support such that

$$\int_{\mathbb{R}} |f - g| < \varepsilon.$$

2. Let $f \in L^1(\mathbb{R})$ and define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = e^{-nx^2} f(x), \quad n \geq 1.$$

Show that $f_n \in L^1(\mathbb{R})$ for each n, and find $\lim_{n\to\infty} \int_{\mathbb{R}} f_n$.

Real Analysis, Optional Problems

3. Suppose that f is a real-valued continuous function on $\mathbb{R}$. Show that the curve

$$\Gamma = \{(x, f(x)) : x \in \mathbb{R}\}$$

has measure zero.

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous. Show that

   (a) f maps sets of measure zero to sets of measure zero.

   (b) f maps measurable sets to measurable sets.

Complex Analysis, Mandatory Problems

1. Compute the following integral:

$$\int_0^\infty \frac{\cos(\pi x)}{(x^2+1)^2}\,dx$$

   Justify all the steps of the calculation.

2. Suppose that $\{f_n\}$ is a sequence of analytic functions on the unit disk

$$\mathbb{D}_1 := \left\{z \in \mathbb{C} : |z| < 1\right\}$$

   such that $f_n \to f$ uniformly on the set $\gamma_r := \{z \in \mathbb{D}_1 \mid |z| = r\}$, for each $0 < r < 1$. Prove that $\{f_n\}$ converges uniformly on any compact subset of $\mathbb{D}_1$ to a function f and that f is analytic on $\mathbb{D}_1$.

Complex Analysis, Optional Problems

3. This is a problem on Laurent expansions.

   (a). State a theorem on the existence and structure of the Laurent expansion of a function f in an annular region about z_0 given by $\{z \in \mathbb{C} \mid 0 \le r < |z - z_0| < R\}$, for $0 \le r < R \le \infty$.

   (b). Suppose a function f is analytic on the annulus about $z_0 = 0$ with $r = 0$ and $R = 1$. If f satisfies the bound

$$|f(z)| \le \frac{1}{|z|^{\frac{1}{2}}},$$

   then prove that f has a removable singularity at $z_0 = 0$ and that f extends to an analytic function g on the disk $|z| < 1$ with $|g(z)| \le 1$.

4. Suppose that $u \in \mathbb{R}$ is not an integer. Prove the identity:

$$\sum_{n=-\infty}^{\infty} \frac{1}{(u+n)^2} = \frac{\pi^2}{\sin^2(\pi u)}.$$

This may be proved by integrating the function

$$f(z) = \frac{\pi \cot \pi z}{(u+z)^2},$$

over a circle of radius $N + \frac{1}{2}$, centered at the origin, for $N > |u|$.