# Preliminary Examination in Analysis

**June 2025**

## Instructions

- This is a three-hour exam on Advanced Calculus and Real Analysis.
- Please work a total of five problems (four mandatory problems, two from each section, and one optional problem). You *must* work the mandatory problems from each part.
- Please indicate clearly on your test paper which optional problem is to be graded.
- Please indicate clearly what theorems and definitions you are using.

# Advanced Calculus, Mandatory Problems

1. Let $(S, d)$ be a compact metric space. Show that if $\{p_j\} \subset S$ is a sequence for which
$$\sum_{j=1}^{\infty} d(p_j, p_{j+1}) < \infty,$$
then $\{p_j\}$ converges to a unique limit in $S$.

2. Suppose the functions $f_n, f : [0,1] \to \mathbb{R}$ are continuous and $f_n \to f$ uniformly on $[0,1]$. Show that
$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx.$$
Provide a counterexample if $f_n \to f$ pointwise but not uniformly.

# Advanced Calculus, Optional Problems

3. Let $\{a_m\}$ be a sequence of real numbers with $\lim_{m\to\infty} a_m = L < \infty$. Let $S_n := \sum_{j=1}^{N} a_j$ be the $N^{th}$-partial sum of the sequence. Prove that
$$\lim_{N\to\infty} \frac{1}{N} S_N = L.$$

4. Suppose that $\{a_j\}$ and $\{b_j\}$ are two bounded sequences of real numbers and that the sequence $\{a_j\}$ converges. Prove that
$$\limsup_{m\to\infty}(a_m + b_m) = \lim_{j\to\infty} a_j + \limsup_{k\to\infty} b_k.$$

# Real Analysis, Mandatory Problems

1. This problem concerns measurable sets in $\mathbb{R}^d$.

   (a) Say what it means for a subset $E \subset \mathbb{R}^d$ to be measurable.

   (b) Suppose that $E \subset \mathbb{R}^d$ is a measurable set. Let $h \in \mathbb{R}^d$ and define

   $$E + h = \{e + h : e \in E\}.$$

   Show that $E + h$ is measurable, and that $m(E + h) = m(E)$.

2. This problem concerns real-valued integrable functions $f : \mathbb{R}^n \to \mathbb{R}$.

   (a) Suppose that $\int_E f(x)\, dx \geq 0$ for every measurable subset $E$ of $\mathbb{R}^n$. Show that $f(x) \geq 0$ a.e.

   (b) Using your result from part (a), show that, if $\int_E f(x)\, dx = 0$ for every measurable subset $E$ of $\mathbb{R}^n$, then $f(x) = 0$ a.e.

# Real Analysis, Optional Problems

3. Let $\{f_k\}$ be a sequence of nonnegative measurable functions on $\mathbb{R}$. Suppose that $f_k \to f$ pointwise a.e., and $f_k \leq f$ a.e. Show that

   $$\lim_{k \to \infty} \int_{\mathbb{R}} f_k\, dx = \int_{\mathbb{R}} f\, dx.$$

4. Suppose $|f|^2$ is integrable on $\mathbb{R}^d$ and

   $$E_\alpha = \{x \in \mathbb{R}^d :\ |f(x)| > \alpha\}.$$

   Show that

   $$\int_{\mathbb{R}^d} |f|^2 = 2 \int_0^\infty \alpha m(E_\alpha)\, d\alpha.$$