Preliminary Examination in Numerical Analysis

Jan, 2003

Instructions:

1. The examination is for 3 hours.
2. The examination consists of two parts:
   Part I: Matrix Theory and Numerical Linear Algebra
   Part II: Numerical Analysis
3. There are three problem sets in each part. **Work two out of the three problem sets for each part.**
4. All problems carry equal weights.

PART I - Matrix Theory and Numerical Linear Algebra
(Work two of the three problems in this part)

Problem 1.

(a) Assume that A and $A+\delta A$ are $n\times n$ invertible matrices and $\eta \equiv \kappa(A)\frac{\|\delta A\|}{\|A\|} < 1$. Prove that

$$\frac{\|(A+\delta A)^{-1} - A^{-1}\|}{\|A^{-1}\|} \leq \frac{\kappa(A)\frac{\|\delta A\|}{\|A\|}}{1-\eta}$$

where $\|\cdot\|$ is any matrix operator norm and $\kappa(A)$ is the condition number of A.

(b) For $A \in R^{n\times n}$, if all its leading principal submatrices $A(1:j, 1:j)$ (for all $1 \leq j \leq n$) are nonsingular, prove by induction on n that the LU factorization of A exists.

Problem 2.

(a) Write down the QR algorithm (unshifted) and describe its convergence properties.

(b) Show that the Hessenberg form is preserved by the QR algorithm (an illustration using a general 4×4 Hessenberg matrix will be sufficient);

(c) Let β be an approximate eigenvalue and $x \in R^n$ with $\|x\|_2 = 1$ a corresponding approximate eigenvector of an $n\times n$ real symmetric matrix A. Prove

$$\min_i |\lambda_i - \beta| \leq \|Ax - \beta x\|_2.$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of A.

Problem 3.

(a) Let A be an $m\times n$ matrix with $\text{rank}(A) = n \leq m$ and $b \in R^n$. If x is the solution to the least squares problem $\min \|Ax - b\|_2$ and $\hat{x}$ is the solution to a perturbed problem $\min \|Ax - b - \delta b\|_2$, prove that

$$\hat{x} - x = (A^T A)^{-1} A^T \delta b.$$

and then

$$\|\hat{x} - x\|_2 \leq \sigma_n(A)^{-1} \|\delta b\|_2$$

where $\sigma_n(A)$ is the smallest singular value of A.

(b) Let A be an $m\times n$ full rank matrix with $m > n$ and let

$$A = QR = Q\begin{pmatrix} R_1 \\ 0 \end{pmatrix}$$

be its QR-decomposition (where R_1 is $n\times n$). For an $m\times k$ matrix B, derive a method to solve

$$\min_{X\in R^{n\times k}} \|AX - B\|_F$$

using the given QR-decomposition.

Part II – Numerical Analysis

(Work two of the three problems in this part)

Problem 4.

(a) State Newton's method to find a solution of the system

$$\begin{aligned} f_1(x_1, x_2) &= 0 \\ f_2(x_1, x_2) &= 0. \end{aligned}$$

(b) Apply one step of the method to the system

$$\begin{aligned} 4x_1^2 - x_2^2 &= 0 \\ 4x_1x_2^2 - x_1 &= 1 \end{aligned}$$

starting at $(1, 0)$.

(c) State the contractive mapping theorem on $[a, b]$ of the real line.

(d) Show that the functional iteration $x_{n+1} = cos(x_n)$ converges for any starting point x_0 to a unique value which is positive.

(e) For the iteration in (d), prove that the order of convergence is linear.

Problem 5. This problem is concerned with interpolation, polynomial approximation, and numerical integration.

(a) Let $h = 1/n$, for a positive integer n, and $x_i = ih$, $i = 0, 1, \cdots, n$. Given $u \in C^2[0, 1]$, let $\widehat{u}$ be the linear spline interpolation of u at $x_0, x_1, \cdots, x_n$. That is, $\widehat{u}$ is piecewise linear and $\widehat{u}(x_i) = u(x_i)$, $i = 0, 1, \cdots, n$. Show that

$$|u(x) - \widehat{u}(x)| \leq \frac{h^2}{8} \max_{\xi \in (0,1)} |u''(\xi)|.$$

(b) Let f and g be two cubic polynomials such that $f(x) = g(x)$ at three distinct points: a, $a + \xi$, and $a + 2\xi$, where $\xi \neq 0$. Prove that

$$\int_a^{a+2\xi} [f(x) - g(x)]\, dx = 0,$$

using the Simpson's Rule with the error term.

Problem 6. Multistep method is used to solve the initial value problem, $y' = f(t, y)$ with $y(0) = a$.

(a) Determine α, β such that the following linear multistep method has order 2 ($f_n = f(t_n, y_n)$ and $h = \Delta t$).

$$y_{n+1} = y_n + h[\alpha f_n + \beta f_{n-1}]$$

(b) Determine the convergence of the method (using characteristic polynomial).