Preliminary Examination in Numerical Analysis

Jan. 9, 2004

Instructions:

1. The examination is for 3 hours.
2. The examination consists of two parts:
   Part I: Matrix Theory and Numerical Linear Algebra
   Part II: Introductory Numerical Analysis
3. There are three problem sets in each part. **Work two out of the three problem sets for each part.**
4. All problem sets carry equal weights.

PART I - Matrix Theory and Numerical Linear Algebra

(Work two of the three problem sets in this part)

Problem 1.

(a) Let A and δA be $n \times n$ matrices and let A be invertible. If $\eta \equiv \kappa(A)\frac{\|\delta A\|}{\|A\|} < 1$, prove that $A + \delta A$ is invertible. Furthermore, if $Ax = b$ and $(A + \delta A)\hat{x} = b$, prove that

$$\frac{\|x - \hat{x}\|}{\|x\|} \leq \frac{\kappa(A)}{1 - \eta} \frac{\|\delta A\|}{\|A\|}$$

where $\|\cdot\|$ is any matrix operator norm and $\kappa(A)$ is the condition number of A. (You may use without proof that $\|(I - X)^{-1}\| \leq (1 - \|X\|)^{-1}$ if $\|X\| < 1$.)

(b) Let $A \in R^{n\times n}$ be symmetric positive definite. Prove by induction on n that A has a Cholesky factorization, namely there exists a lower triangular matrix G such that $A = GG^T$.

Problem 2.

(a) Write down the QR algorithm (unshifted) for an $n \times n$ matrix A. Prove that the matrices produced are all similar to the original matrix.

(b) Describe an algorithm to reduce a symmetric matrix to a tridiagonal matrix through a sequence of orthogonal similarity transformations.

(c) Show that the tridiagonal form is preserved by the QR algorithm (an illustration using a 4×4 tridiagonal matrix will be sufficient).

Problem 3. Let $A \in R^{m\times n}$ and $b \in R^m$ $(m \geq n)$. Let $A = U\Sigma V^T$ be the singular value decomposition of A, where

$$\Sigma := \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \in R^{m\times n}; \;\; \Sigma_1 := \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{pmatrix}$$

with $\sigma_1 \geq \cdots \geq \sigma_k > 0$ is $k \times k$.

(a) Determine when $Ax = b$ has no solution, exactly one solution, or infinitely many solutions. Write down the solution or the solution set when it exists.

(b) Determine when the least squares problem

$$\min_{x\in R^n} \|Ax - b\|_2. \tag{1}$$

has exactly one solution, or infinitely many solutions. Write down the solution or the solution set when it exists.

Part II – Numerical Analysis
(Work two of the three problem sets in this part)

Problem 4. Suppose $g(x)$ is a C^1 function with a fixed point z, i. e. $g(z) = z$, and

$$|g'(z)| = \alpha \quad < 1$$

(a) Prove that a fixed point iteration will converge linearly to z from any point x_0 sufficiently close to z.

(b) What is the rate of convergence ?

(c) Perform one iteration of Newton's method on the system:

$$\begin{aligned} x_1^2 - 2x_1 - x_2 + 0.5 = 0 \\ x_1^2 + 4x_2^2 - 4 = 0 \end{aligned}$$

starting at point $(2, 0.25)$.

Problem 5. Outline the ideas and steps to derive a Gauss Formula

$$\int_{-1}^{1} f(x)dx = \sum_{i=0}^{n} A_i w_i f(x_i)$$

which is exact for all the polynomials of degree ≤ 3 on $[-1, 1]$.

(a) How many nodes (minimum number) are needed for Gauss Formula to be exact for all the polynomials of degree ≤ 3, i.e., what is n? and why?

(b) Use a theorem about orthogonal polynomials and the fact that 1, x, $x^2 - \frac{1}{3}$, $x^3 - \frac{3}{5}x$ are orthogonal on $[-1, 1]$ with weight function $w_i = 1$ to determine x_i.

(c) Use method of undetermined coefficient to find A_i and write the Gauss Formula.

Problem 6. Where $x'(t) = f(t, x)$, $x(0) = x_0$ and $f_n = f(t_n, x_n)$, the formula

$$x_{n+1} - (1 - c)x_n - cx_{n-1} = \frac{h}{12}[(5 - c)f_{n+1} + 8(1 + c)f_n + (5c - 1)f_{n-1}]$$

is known to be exact for all polynomials of degree m or less for all c.

(a) Determine c so that it will be exact for all polynomials of degree $m + 1$. Find c and m.

(b) Using the c found in (a), is this method stable? strongly stable? is this method consistent? convergent?