Preliminary Examination in Numerical Analysis

January 2005

Instructions:

- Examination is 3 hours: 9am to noon
- Part I: Numerical Linear Algebra
- Part II: Numerical Analysis
- Each part has 3 problems, and 6 problems total. You must do Problems 2 and 4, but have a choice of solving only 3 of the remaining 4 problems. Mark the problems that you wish to be graded in the table below (5 marks total).
- For problems with multiple parts, you may get full credit for a particular part using results of previous parts even though you fail to finish these previous parts

Problem No.	to be graded?	grade
1		
2	$\surd$	
3		
4	$\surd$	
5		
6		
Total		

Part I. Numerical Linear Algebra

Problem 1. Least Square Problem. Let $A \in \mathbb{R}^{m\times n}$ and $m > n$. Outline at least two different numerical methods to solve

$$\min_x \|Ax - b\|_2,$$

and explain briefly their pros and cons.

Problem 2. Eigenvalue Computation. Let $A \in \mathbb{R}^{n\times n}$.

1. State the power method;
2. Present a convergence analysis of the power method, assuming that A is diagonalizable;
3. Is it possible to use this method to compute an eigenvalue that is near any given value μ? If not, what would you do?
4. In MATLAB, $\mathsf{eig}(A)$ outputs all eigenvalues of A. How does it do that?

Problem 3. Orthogonal Projections.

1. Let $X \in \mathbb{R}^{m\times n}$ have full column rank. The orthogonal projection onto X's column space is $P_X = X(X^TX)^{-1}X^T$, and $P_X^\perp = I - P_X$ is the orthogonal projection onto the orthogonal complement of X's column space. Verify that $P_X X = X$ and $P_X^\perp X = 0$.

2. Consider $Z_1 = \begin{matrix} & n \\ m_{11} & \\ m_{12} & \end{matrix}\!\begin{pmatrix} Z_{11} \\ Z_{12} \end{pmatrix}$, $Z_2 = \begin{matrix} & n \\ m_{21} & \\ m_{22} & \end{matrix}\!\begin{pmatrix} Z_{21} \\ Z_{22} \end{pmatrix}$, $Z = \begin{matrix} & n \\ m_{11} & \\ m_{21} & \\ m_{22} & \end{matrix}\!\begin{pmatrix} Z_{11} \\ Z_{21} \\ Z_{22} \end{pmatrix}$, where $m_{12} = m_{21}$, and $Z_{12} = Z_{21}$ is the common part in Z_1 and Z_2. Set

$$P = \begin{matrix} & m_{11}+m_{12} & m_{22} \\ m_{11}+m_{12} & & \\ m_{22} & & \end{matrix}\begin{pmatrix} P_{Z_1}^\perp & 0 \\ 0 & 0 \end{pmatrix} + \begin{matrix} & m_{11} & m_{12}+m_{22} \\ m_{11} & & \\ m_{12}+m_{22} & & \end{matrix}\begin{pmatrix} 0 & 0 \\ 0 & P_{Z_2}^\perp \end{pmatrix},$$

Show that $PZ = 0$.

Part II. Numerical Analysis

Problem 4.

1. Consider a variation of Newton's method in which only one derivative is needed: that is,
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}$$
Find C and s such that $e_{n+1} = Ce_n^s$, assuming it converges, where $e_n = x_n - x_*$ and x_* is the desired solution to $f(x) = 0$.

2. Suppose that the bisection method is started with the interval $[10, 35]$. How many steps should be taken to compute a root with relative accuracy no bigger than 10^{-12}?

3. Suppose we are solving $\phi(z) = z$ for z, where $\phi : \mathbb{R}^n \to \mathbb{R}^n$; and suppose $\|\phi(x) - z\| \leq \|x - z\|^2$ for $\|x - z\| \leq 1$. Show that there exists $\delta > 0$ such that if $\|x_0 - z\| < \delta$, then the sequence generated by $x_{n+1} = \phi(x_n)$ converges to z quadratically (i.e. order 2). (You may use the usual contraction mapping theorem.)

Problem 5.

1. Write the Newton interpolating polynomial $p_3(x)$ which interpolates the function $f(x) = 2\sin(\frac{\pi}{3}x)$ at points $x = 0, 1, 2$ and 5.

2. (continuing 1) what is a good upper bound for $|f(x) - p_3(x)|$ on $[0, 5]$.

3. Prove that if f is a polynomial of degree k, then for $n > k$,
$$f[x_0, x_1, \ldots, x_n] = 0.$$
(Newton's divided difference formula).

4. Find a Least-Squares fit of the form $y = Ax^3$, for the following data set, $f(-1) = -2, f(0) = -1, f(1) = 4, f(2) = 7$.

Problem 6.

1. What is the order of accuracy for Backward Euler method?

2. What is the absolute stability region for Backward Euler method? (Consider the Cauchy Problem $y' = \lambda y$, $y(0) = 1$.)

3. Derive the second-order Rung-Kutta formula

$$x(t+h) = x(t) + \frac{1}{2}(F_1 + F_2),$$

$$\begin{cases} F_1 = hf(t, x), \\ F_2 = hf(t+h, x+F_1). \end{cases}$$