Preliminary Examination in Numerical Analysis

Jan. 4, 2006

Instructions:

1. The examination is for 3 hours.
2. The examination consists of two parts:
   Part I: Matrix Theory and Numerical Linear Algebra
   Part II: Introductory Numerical Analysis
3. There are three problem sets in each part. **Work two out of the three problem sets for each part.**
4. All problem sets carry equal weights.

PART I - Matrix Theory and Numerical Linear Algebra

(Work two of the three problem sets in this part)

Problem 1. Let $\mathrm{fl}\,(x)$ denote computational result of an expression x in a floating point arithmetic and let ϵ be the machine roundoff unit.

1. Show that
$$\mathrm{fl}\,(\sum_{i=1}^{n} x_i y_i) = \sum_{i=1}^{n} x_i y_i (1+\delta_i)$$
with $\delta_i \le n\epsilon + \mathcal{O}(\epsilon^2)$.

2. Let $A,\ B \in R^{n\times n}$. Show that
$$\mathrm{fl}\,(AB) = AB + E, \quad |E| \le n\epsilon |A|\,|B| + \mathcal{O}(\epsilon^2).$$

3. Let $L = [l_{ij}]$ be an $n \times n$ lower triangular matrix with the diagonals equal to 1. For any $b \in R^n$, consider solving $Lx = b$ by the forward substitution
$$x_i = b_i - \Sigma_{j=1}^{i-1} l_{ij} x_j, \quad i = 1, \cdots, n.$$
Prove that the computed solution $\hat{x}$ satisfies $(L + \delta L)\hat{x} = b$ with $|\delta L| \le (n-1)\epsilon |L| + \mathcal{O}(\epsilon^2)$.

Problem 2.

1. Let $\hat{x}$ be an approximate solution to $Ax = b$ and $r = A\hat{x} - b$. Construct a matrix δA such that $(A + \delta A)\hat{x} = b$ and
$$\frac{\|\delta A\|_2}{\|A\|_2} = \frac{\|r\|_2}{\|A\|_2 \|\hat{x}\|_2}.$$

2. Let $A = LU$ be the LU-factorization of A with $|l_{ij}| \le 1$. Let a_i^T and u_i^T denote the i-th row of A and U respectively. Show that
$$u_i^T = a_i^T - \Sigma_{j=1}^{i-1} l_{ij} u_j^T.$$
and
$$\|U\|_\infty \le 2^{n-1} \|A\|_\infty$$

3. Let $A \in R^{m\times n}$ and $b \in R^m$ $(m \ge n)$. Outline the algorithm to compute the QR factorization of A using the Householder transformations and the method to solve $\min_{x\in R^n} \|Ax - b\|_2$ by the QR factorization.

Problem 3. Eigenvalue Problem.

1. Consider the orthogonal iteration for $A \in R^{n\times n}$:

Algorithm: $V_0 = I$;
For $i = 0, 1, 2, \cdots, m$
  $Y_i = AV_i$
  Factorize $Y_i = V_{i+1}\hat{R}_{i+1}$ (QR-factorization)
End

Prove that $A^i = V_i \hat{R}_i \hat{R}_{i-1} \cdots \hat{R}_1$. Describe the convergence property of the columns of V_i (make necessary assumptions).

2. Consider the QR algorithm:

Algorithm: $A_0 = A$
For $i = 0, 1, 2, \cdots, m$
  Factorize $A_i = Q_i R_i$ (QR-factorization)
  $A_{i+1} = R_i Q_i$
End

Prove that $A_{i+1} = Q_i^T A_i Q_i$.

3. Prove that the two algorithms are equivalent in the sense that $A_i = V_i^T A V_i$.

Part II – Numerical Analysis
(Work two of the three problem sets in this part)

Problem 4.

1. Denote the intervals that arise in the bisection method by $[a_0, b_0], [a_1, b_1], \ldots, [a_n, b_n]$. Let the midpoint of each interval be $c_n = (a_n + b_n)/2$, $r = \lim_{n \to \infty} c_n$, and $e_n = r - c_n$. For the following (a) and (b), if the statement is true prove it, and if the statement is false show a simple counter example.

   (a) $|r - a_n| \leq 2^{-n-1}(b_o - a_o)$

   (b) $0 \leq r - a_n \leq 2^{-n}(b_o - a_o)$

   (c). Suppose that the bisection method is started with the interval $[0.1, 1.0]$. How many steps should be taken to computer a root with relative accuracy of $\frac{1}{2} \times 10^{-8}$?

2. Determine the formula for Newton's method for finding a root of the function $f(x) = x - e/x$. Prove the order of convergence for this Newton's method.

Problem 5.

1. (a) Find the Chebyshev polynomial interpolation $p(x)$ that approximates the function $f(x) = \dfrac{1}{1+x^2}$ over $[-1, 1]$ with $n = 2$.

   (b) What is the error bound for$|f(x) - p(x)|$?

   (c) What is the error bound if one use any three points to interpolate $f(x)$?

2. Establish a formula of the form

$$f''(x) \sim \frac{1}{h^2}[Af_{n+2} + Bf_{n+1} + Cf_n + Df_{n-1}]$$

   where $f_{n+1} = f(x_n + h)$. What is the order of error for this approximation ?

Problem 6.

1. Find an equation of the form $y = ae^x + bx$ that best fits the points $(-1, 0)$, $(0, 1)$ and $(1, 2)$ in the least-squares sense.

2. Discuss the convergence of the following method for solving $x' = f(t, x)$ with $x(0) = x_o$,

$$x(t + h) = x(t) + \frac{1}{2}(K_1 + K_2)$$

   where

$$\begin{aligned} K_1 &= hf(t, x) \\ K_2 &= hf(t + h, x + K_1) \end{aligned}$$