Preliminary Examination in Numerical Analysis

Jan. 4, 2007

Instructions:

1. The examination is for 3 hours.
2. The examination consists of two parts:
   Part I: Matrix Theory and Numerical Linear Algebra
   Part II: Introductory Numerical Analysis
3. There are three problem sets in each part. **Work two out of the three problem sets for each part.**
4. All problem sets carry equal weights.
5. Problems within each problem set are not necessarily related.

PART I - Matrix Theory and Numerical Linear Algebra
(Work two of the three problem sets in this part)

Problem 1.
Let X and Y be invertible matrices with $\|X - Y\| \leq \epsilon$ and $\epsilon\|X^{-1}\| < 1$.

a) Show that $\|Y^{-1}\| \leq \dfrac{\|X^{-1}\|}{1 - \epsilon\|X^{-1}\|}$.

b) Show that $\dfrac{\|X^{-1} - Y^{-1}\|}{\|X^{-1}\|} \leq \kappa(X)\dfrac{\epsilon/\|X\|}{1 - \epsilon\|X^{-1}\|}$, where $\kappa(X)$ is the condition number of X.

c) Show that equality holds in (b) for the matrices

$$\begin{aligned} X &= \begin{bmatrix} a & 0 \\ 0 & \frac{1}{a} \end{bmatrix} \\ Y &= X + \begin{bmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{bmatrix} \end{aligned}$$

when $0 < \epsilon < a < 1$.
(Hint: Consider $Y^{-1}(X - Y)X^{-1}$.)

Problem 2.

a) Let v be a given $n \times 1$ real unit vector. Use a Householder reflection to define an $n \times n$ orthogonal matrix V with v as its first column.

b) Show that an orthogonal matrix that is also upper triangular must be diagonal.

c) Let H and K be upper Hessenberg matrices where H has nonzero subdiagonal entries and suppose Q is an orthogonal matrix with $K = QHQ^T$ and $Qe_1 = e_1$. Show that Q is a diagonal matrix.

Problem 3. Let A be an $m \times n$ matrix with singular value decomposition $A = U\Sigma V^T$ and put $H = \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix}$. For $i = 1, \ldots, n$, let σ_i be the i th diagonal element of Σ and let u_i and v_i be the i th columns of U and V, respectively. $m + n$

a) Prove that σ_i and $-\sigma_i$ are eigenvalues of H with corresponding eigenvectors $\begin{bmatrix} v_i \\ u_i \end{bmatrix}$ and $\begin{bmatrix} v_i \\ -u_i \end{bmatrix}$, respectively.

b) Use part (a) to define an orthogonal matrix Q and a diagonal matrix D so that $H = QDQ^T$.

c) Let $b \in R^m$. If x is a solution to the least squares problem $\min_{x \in R^n} \|b - Ax\|_2$, prove that $z = \begin{bmatrix} x \\ r \end{bmatrix}$ is a solution to

$$\begin{bmatrix} 0 & A^T \\ A & I \end{bmatrix} z = \begin{bmatrix} 0 \\ b \end{bmatrix},$$

for some $r \in R^m$

Part II – Numerical Analysis
(Work two of the three problem sets in this part)

Problem 4.

1. Use the definition of contractive mapping to prove the following theorem.
   [Contractive Mapping Theorem] Let C be a closed subset of the real line. If F is a contractive mapping of C into C, then F has a unique fixed point. Moreover, this fixed point is the limit of every sequence obtained from $x_{n+1} = F(x_n)$ with a starting point $x_0 \in C$.

2. Show that the iteration
$$x_{k+1} = \cos(x_k)$$
   converges to the fixed point $\xi = \cos(\xi)$ for all $x_0 \in \mathcal{R}$.

Problem 5.

1. Find the (a) Lagrange interpolation polynomial and (b) Newton's polynomial interpolating the following data,

x	0	1	2	3
f(x)	30	6	-2	-3

2. Prove the following theorem on polynomial interpolation error.
   Let f be a function in $C^{n+1}[a,b]$, and let p be the polynomial of degree at most n that interpolates the function f at $n+1$ distinct points $x_0, x_1, \ldots, x_n$ in the interval $[a,b]$. To each x in $[a,b]$ there corresponds a point ξ_x in (a,b) such that
$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x)\Pi_{i=0}^{n}(x - x_i).$$

Problem 6.

1. (a) Determine the values of $a, b, c,$ and d so that the following is a natural cubic spline:
$$f(x) = \begin{cases} 3 + x - 9x^3 & x \in [0,1] \\ a + b(x-1) + c(x-1)^2 + d(x-1)^3 & x \in [1,2] \end{cases}$$
   (b) Determine the values of $a, b, c,$ and d so that $f(x)$ above is a cubic spline with $\int_0^2 [f''(x)]^2 dx$ minimized.

2. Determine a formula of the form
$$\int_{-1}^{1} f(x)dx \approx w_0 f(-\frac{1}{2}) + w_1 f(0) + w_2 f(\frac{1}{2})$$
   that is exact for all polynomials of as high a degree as possible. What is the highest degree?