Preliminary Examination in Numerical Analysis

January 6, 2010

Instructions:

1. The examination is for 3 hours.
2. The examination consists of two parts:
   Part I: Matrix Theory and Numerical Linear Algebra
   Part II: Introductory Numerical Analysis
3. There are three problem sets in each part. **Work two out of the three problem sets for each part.**
4. All problem sets carry equal weights (but not necessarily the various problems within each problem set).
5. Problems within each problem set are not necessarily related but they may be. You could use the result from one part in your solutions for other parts, even if you did not prove it.

PART I - Matrix Theory and Numerical Linear Algebra
(Work two of the three problem sets in this part)

Problem 1. Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ $(m \geq n)$. Let $A = U\Sigma V^T$ be the singular value decomposition of A, where

$$\Sigma := \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{m\times n}; \quad \Sigma_1 := \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{pmatrix}$$

with $\sigma_1 \geq \cdots \geq \sigma_k > 0$.

(a) Determine when $Ax = b$ has no solution, exactly one solution, or infinitely many solutions. Write down the solution or the solution set in terms of singular vectors when it exists.

(b) Determine when the least squares problem

$$\min_{x\in R^n} \|Ax - b\|_2. \tag{1}$$

has exactly one solution, or infinitely many solutions. Write down the solution or the solution set in terms of singular vectors when it exists.

Problem 2.

1. Let $A \in \mathbb{C}^{n\times n}$ and $x \in \mathbb{C}^n$ with $\|x\|_2 = 1$ be given, and define $\mu \in \mathbb{C}$ by $\mu = x^H Ax$—think of (μ, x) as an approximate eigenpair for A. Note: the superscript ^{H} denotes the conjugate transpose of the matrix or vector to which it is applied.

   (a) Show that, for any $w \in \mathbb{C}^n$ with $\|w\|_2 = 1$, $M = I - 2ww^H$ is Hermitian and unitary. Let $e_k \in \mathbb{R}^n$ be the k^{th} column of the of the identity matrix $I \in \mathbb{R}^{n\times n}$, and $\beta \in \mathbb{C}$ have unit modulus, $|\beta| = 1$. Show that $D = I - (1+\beta)e_k e_k^H$ is diagonal and unitary.

   (b) Give a constructive proof that there exists a unitary matrix $Q \in \mathbb{C}^{n\times n}$ having x as its first column, i.e. $Qe_1 = x$.

   (c) Define the vector $e \in \mathbb{C}^{n-1}$ via the following block partitioning of $Q^H AQ$,

   $$Q^H AQ = \begin{pmatrix} \mu & b^H \\ e & C \end{pmatrix}.$$

   Show that $\|e\|_2 = \|Ax - \mu x\|_2 = \min_{\sigma\in\mathbb{C}} \|Ax - \sigma x\|_2$.

Problem 3.

1. Consider multiplying $A = [a_{ij}] \in R^{n\times n}$ with a vector $x \in R^n$ in a floating point arithmetic. Prove that
$$fl(Ax) = (A + \delta A)x$$
for some δA with $|\delta A_{ij}| \le n|a_{ij}|(\epsilon + \mathcal{O}(\epsilon^2))$, where ϵ is the machine roundoff unit. (You may use $fl(\sum_{i=1}^{d} x_i y_i) = \sum_{i=1}^{d} x_i y_i (1 + \delta_i)$ with $\delta_i \le d\epsilon + \mathcal{O}(\epsilon^2)$.)

2. Let A be an $n \times n$ symmetric positive definite matrix. If E is a symmetric matrix such that $\|E\|_2 < \|A^{-1}\|_2^{-1}$, prove that $A + E$ is symmetric positive definite.

3. Write down the shifted QR algorithm for computing eigenvalues of an $n \times n$ matrix A. Prove that the matrices produced are all similar to the original matrix.

Part II – Numerical Analysis
(Work two of the three problems in this part)

Problem 4.

(a) Choosing a, b, and $\frac{a+b}{2}$ as quadrature points, use the method of undetermined coefficients to derive a quadrature method for approximating $\int_a^b f(x)dx$. Show that your method is exact for all cubic polynomials, but not generally for quartics.

(b) Suppose that you wish to numerically solve the initial value problem $x'(t) = f(x,t)$, $x(0) = x_0$. The modified Euler's method

$$x(t+h) = x(t) + hf(t + \frac{1}{2}h, x(t) + \frac{1}{2}hf(t, x(t)))$$

is a Runge-Kutta scheme for this IVP. Use Richardson extrapolation on Euler's method with step sizes h and $\frac{h}{2}$ in order to derive the modified Euler method. Recall that Euler's method is given by $x(t+h) \approx x(t) + hx'(t)$. You may assume for simplicity that the error term in Euler's method is Kh^2, that is, $x(t+h) = x(t) + hx'(t) + Kh^2$, with K independent of x.

Problem 5.

(a) Show that the error term for the simple Midpoint Rule on an interval $[a,b]$ is $\frac{1}{24}(b-a)^3 f''(c)$, where $c \in [a,b]$. More precisely, show that if f is sufficiently smooth, then

$$\int_a^b f(x)dx = (b-a)f(\frac{a+b}{2}) + \frac{1}{24}f''(c)(b-a)^3$$

for some $c \in [a,b]$. You may want to use the Integral Mean Value Theorem: Assume that u and v are continuous on $[a,b]$, and that $v \geq 0$. Then there exists $\xi \in [a,b]$ such that $\int_a^b u(x)v(x)dx = u(\xi)\int_a^b v(x)dx$.

(b) Suppose that you enter any number into a calculator and repeatedly press the cos button. Will the sequence of numbers that is thus generated converge? Be sure to provide a proof for your answer. (You may assume that the calculator has infinite precision, so that floating point errors can be ignored.)

(c) Find the Newton form of the polynomial p of lowest possible degree satisfying $p(0) = 1$, $p(1) = 2$, $p(2) = -1$, and $p'(1) = 0$.

Problem 6.

(a) Assume that $\{x_n\}$ is generated by Newton's method for a function f, where f'' is continuous, $f(r) = 0 \neq f'(r)$, and x_0 is sufficiently close to r. Show that

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n^2} = \frac{f''(r)}{2f'(r)}.$$

(b) Let

$$f(x, y) = \begin{bmatrix} y^2 - 1 \\ x^2 - 2 \end{bmatrix}.$$

Carry out one step of Newton's method for f with starting point $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

(c) Consider the following numerical scheme for solving $x' = f(x, t)$:

$$x_n - 3x_{n-1} + 2x_{n-2} = h[f_n + 2f_{n-1} + f_{n-2} - 2f_{n-3}].$$

Using the notions of consistency, stability, and convergence, discuss whether this would be a good scheme to use in practice.