Preliminary Examination in Numerical Analysis

January 5, 2011

Instructions:

1. The examination is for 3 hours.
2. The examination consists of two parts:
   Part I: Matrix Theory and Numerical Linear Algebra
   Part II: Introductory Numerical Analysis
3. There are three problem sets in each part. **Work two out of the three problem sets for each part.**
4. All problem sets carry equal weights.
5. Problems within each problem set are not necessarily related but they may be. You may use the results from one part in your solutions for other parts, even if you did not prove it.

PART I - Matrix Theory and Numerical Linear Algebra

(Work two of the three problem sets in this part)

Problem 1. Let $\mathrm{fl}\,(x)$ denote the computational result of an expression x in a floating point arithmetic and let ϵ be the machine roundoff unit. Let $|A|$ denote the matrix obtained from A by applying absolute values in each entry.

(a) Let $A,\ B \in R^{n\times n}$. Prove that

$$\mathrm{fl}\,(AB) = AB + E, \quad |E| \le \big(n\epsilon + \mathcal{O}(\epsilon^2)\big)\,|A|\,|B|.$$

You may use without proof that

$$\mathrm{fl}\,(\sum_{i=1}^{n} x_i y_i) = \sum_{i=1}^{n} x_i y_i (1+\delta_i)$$

with $|\delta_i| \le n\epsilon + \mathcal{O}(\epsilon^2)$.

(b) Let $A \in R^{n\times n}$ be symmetric positive definite. Prove by induction on n that A has a Cholesky factorization.

Problem 2.

(a) Describe the algorithm of using the Householder transformation to compute the QR factorization of an $m \times n$ matrix A.

(b) Let $A \in R^{m\times n}$ and $b \in R^m$ $(m \ge n)$. Assume that A has full column rank and $A = QR$ is the QR factorization of A. Prove that $x = R^{-1}Q^T b$ is the solution to the least squares problem

$$\min_{x\in R^n} \|Ax - b\|_2.$$

Problem 3.

(a) Let $A \in R^{m\times m}$ and $B \in R^{n\times n}$ be normal matrices. Let $C \in R^{m\times n}$. Prove that

$$X = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$$

is normal if and only if $C = 0$.

(b) Write down the QR algorithm (unshifted) for computing the Schur form of an $n \times n$ matrix A. Prove that the matrices produced are all similar to the original matrix.

(c) Show that the Hessenberg form is preserved by the QR algorithm (an illustration using a 4×4 tridiagonal matrix will be sufficient).

Part II – Numerical Analysis
(Work two of the three problem sets in this part)

Problem 4.

a) Given below are partial convergence histories of the absolute error, $|x_n - \sqrt{2}|$, of four methods for approximating $\sqrt{2}$: the bisection method, Newton's method and the secant method on the function $f(x) = x^2 - 2$; and fixed-point iteration with the function $g(x) = (x+2)/(x+1)$. Assign each of the columns with the appropriate label: "Bisection", "Newton", "Secant", "Fixed-Point".

4.8596e+01	6.7080e-13	1.8405e-09	4.9351e+00
2.3611e+01	2.1605e-13	3.1577e-10	2.7468e+00
1.1138e+01	1.1324e-14	5.4178e-11	1.2898e+00
4.9417e+00	1.0236e-13	9.2957e-12	5.1606e-01
1.9211e+00	4.5519e-14	1.5947e-12	1.4362e-01
5.5325e-01	1.7097e-14	2.7378e-13	2.1249e-02
7.7787e-02	2.8866e-15	4.6851e-14	1.0196e-03
2.0278e-03	4.2188e-15	8.2157e-15	7.5997e-06
1.4517e-06	6.6613e-16	1.3323e-15	2.7385e-09
7.4496e-13	1.1102e-15	4.4409e-16	7.3275e-15
2.2204e-16	2.2204e-16	0.0000e+00	2.2204e-16

b) Let $g(x) = 1/(1+x)$. Prove that, if $x_0 > 0$, then the iteration defined by $x_{n+1} = g(x_n)$ converges. To what point does it converge?

c) Let $f(x) = x^2$. Prove that Newton's method converges for any starting point x_0. What is the order of convergence if $x_0 \neq 0$?

d) Let $f(x) = (x-1/5)^2(x-1/3)(x-2/3)(x-4/5)$. Suppose that the bisection method is used to find a root of f, with the starting interval $[0, 1]$. To what point does the iteration converge?

Problem 5. The minimal-degree Hermite interpolant of f, satisfying $p(a) = f(a)$, $p'(a) = f'(a)$, $p(b) = f(b)$ and $p'(b) = f'(b)$, can be expressed as

$$\begin{aligned} p(x) = f(a) + f'(a)(x-a) + \left[\frac{f(b)-f(a)}{(b-a)^2} - \frac{f'(a)}{b-a}\right](x-a)^2 \\ + \left[\frac{f'(a)+f'(b)}{(b-a)^2} - 2\frac{f(b)-f(a)}{(b-a)^3}\right](x-a)^2(x-b) \ . \end{aligned}$$

We will assume that f is four times continuously-differentiable on $[a,b]$.

a) Prove that, for each $x \in [a,b]$, there is a $\xi(x) \in [a,b]$ for which

$$f(x) - p(x) = \frac{f^{(4)}(\xi(x))}{4!}(x-a)^2(x-b)^2 \ .$$

Hint: When $x \in (a,b)$, consider the function

$$g(t) = f(t) - p(t) - \frac{f(x)-p(x)}{(x-a)^2(x-b)^2}(t-a)^2(t-b)^2 \ .$$

What are the values of g and g' at $t = a$, $t = b$ and $t = x$?

b) Show that the quadrature error in the in the *corrected trapezoid rule* ,

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(a)-f'(b)] \ ,$$

is given by

$$\int_a^b f(x)\,dx - \left(\frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(a)-f'(b)]\right) = \frac{f^{(4)}(\xi)}{720}(b-a)^5 \ ,$$

for some $\xi \in [a,b]$.

c) Consider the initial value problem

$$x'(t) = \sin(x(t)) \ , \ x(0) = x_0 \ .$$

Using the uniform mesh $t_j = jh$, $j \geq 0$, give the sharpest possible bound (based on the information given) on the *local* truncation error in the approximation $x_j \approx x(t_j)$ defined by

$$x_{n+1} = x_n + \frac{h}{2}[\sin(x_n) + \sin(x_{n+1})] + \frac{h^2}{12}[\cos(x_n)\sin(x_n) - \cos(x_{n+1})\sin(x_{n+1})] \ .$$

Problem 6.

a) Consider the IVP $x'(t) = f(x,t)$, $x(0) = x_0$. Use the method of undetermined coefficients to determine the third-order Adams-Bashforth iteration for the uniform mesh $t_j = jh$, $j \geq 0$:

$$x_{n+1} = x_n + h[Af_n + Bf_{n-1} + Cf_{n-2}] \ ,$$

where $f_j = f(x_j, t_j)$, and $x_j \approx x(t_j)$ is our approximation.

b) The Chebyshev polynomials (of the first kind) are given by $T_n(x) = \cos(n \arccos(x))$ on $[-1,1]$, for $n \geq 0$.

i) Show that $\int_{-1}^{1} T_n(x)T_m(x)(1-x^2)^{-1/2}\,dx = 0$ for $m \neq n$.

ii) Determine the roots x_1, x_2 of T_2.

iii) Find weights w_1, w_2 so that the quadrature

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx \approx w_1 f(x_1) + w_2 f(x_2)$$

is exact for all polynomials of degree 3 or less.

c) Suppose that

x	$f(x)$	$f'(x)$	$f''(x)$
-1	1	-1	4
0	-1	-1	
1	-41		

Give the Hermite interpolant of this data in its Newton form.