Preliminary Examination in Numerical Analysis

Jan. 13, 2014

Instructions:

1. The examination is for 3 hours.
2. The examination consists of two parts, each consisting of six equally-weighted problems:
   Part I: Matrix Theory and Numerical Linear Algebra
   Part II: Introductory Numerical Analysis
3. You may omit one problem from Part I and one problem from Part II.

Problem 1. Consider conditioning of the solution to the system $(\alpha I + A)x = b$ subject to perturbations to α. Assuming $\alpha I + A$ is invertible show that for $\delta\alpha$ sufficiently small

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\|(\alpha I + A)^{-1}\||\alpha|}{1 - \|(\alpha I + A)^{-1}\||\alpha|\frac{|\delta\alpha|}{|\alpha|}} \frac{|\delta\alpha|}{|\alpha|}$$

where δx is the perturbation to the solution, i.e. $((\alpha + \delta\alpha)I + A)(x + \delta x) = b$.

Problem 2. Consider the elementary matrix $U = I - ue_k^T$ where $u \in \mathbb{R}^n$ has zeroes in its k to n entries and $e_k \in \mathbb{R}^n$ is the k-th column of the identity matrix I. Assume that the entries of u and $b \in \mathbb{R}^n$ are machine numbers. Show that the solution $\widehat{x}$ to $Ux = b$ computed in floating point arithmetic using backward substitution satisfies $(U + E)\widehat{x} = b$, where $|E| \leq (\varepsilon + O(\varepsilon^2))|U|$ and ε is the machine precision.

Problem 3. Show that the growth factor of the Gaussian elimination with partial pivoting applied to an $n \times n$ unreduced upper Hessenberg matrix $A = [a_{ij}]$ (i.e., $a_{ij} = 0$ for $i > j + 1$) is bounded above by n.

Problem 4. Suppose we wish to solve the following constrained least squares problem

$$\min_{Cx=d} \|Ax - b\|,$$

where $A \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{s\times n}$, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, and $d \in \mathbb{R}^s$. Assuming that $m > n > s$ and C and A are both full rank, describe how one might solve this problem using QR factorizations.

Problem 5. Describe how the following matrix can be brought to upper triangular form using only seven Givens rotations as opposed to ten,

$$A = \begin{pmatrix} x & x & x & x & x \\ x & x & x & x & x \\ x & & x & x & x \\ x & & & x & x \\ x & & & & x \end{pmatrix}.$$

Show that the same approach for larger matrices with the same sparsity pattern, i.e., upper triangular except in the first column, requires $O(n)$ operations.

Problem 6. Consider the following iteration for a $m \times n$ matrix A (with $m > n$) given some random nonzero starting vector $x_0 \in \mathbb{R}^n$,

$$\begin{aligned} y_i &= \frac{Ax_i}{\|Ax_i\|} \\ x_{i+1} = &= \frac{A^T y_i}{\|A^T y_i\|} \end{aligned}$$

Under what conditions will this iteration converge, at what rate, and what will x_i and y_i converge to? Provide a brief explanation to your answers.

PART 2 - Numerical Analysis

Problem 7. Let F satisfy $|F'(x)| \le \lambda < 1$ on the interval $[x_0 - \rho, x_0 + \rho]$, where $\rho = |F(x_0) - x_0|/(1-\lambda)$. Show that the sequence generated by the iteration $x_{n+1} = F(x_n)$ will converge to $x \in [x_0 - \rho, x_0 + \rho]$.

Problem 8. Assume that $\{x_n\}$ is generated by Newton's method for a function f, where f'' is continuous, $f(r) = 0 \neq f'(r)$, and x_0 is sufficiently close to r. Show that the sequence $\{x_n\}$ converges quadratically to r.

Problem 9. Let $f(x) = \cos(\pi x)$, and let p be the Hermite interpolant to f with nodes $x_0 = -1/2$ and $x_1 = 1/2$. That is, $p(x_i) = f(x_i)$ and $p'(x_i) = f'(x_i)$, $i = 0, 1$. Give a bound for $\max_{-1\le x\le 1} |f(x) - p(x)|$. You may cite a theorem as part of your solution, but your final answer should be a number.

Problem 10. Show that the error term for the Trapezoid Rule on the interval $[a, b]$ is $-\frac{1}{12}(b-a)^3 f''(\xi)$, where $\xi \in [a, b]$. More precisely, show that if f is sufficiently smooth, then

$$\int_a^b f(x)\mathrm{d}x = (b-a)\frac{f(a)+f(b)}{2} - \frac{1}{12}f''(\xi)(b-a)^3.$$

Problem 11. Let $w(x) = x^2$. Find a Gaussian quadrature formula of the form $\int_0^1 w(x)f(x)\mathrm{d}x \approx A_1 f(x_1) + A_2 f(x_2)$ that is exact for all polynomials of degree 3 or less.

Problem 12. Let $\tau_k(h)$ be the truncation error of the approximation method

$$y_{k+1} = y_k + h\phi(t_k, y_k, h)$$

for $y'(t) = f(t, y(t)), y(t_0) = \gamma$. Assume that $|\phi(t, w, h) - \phi(t, v, h)| \le L|w - v|$ for some $L > 0$. If $\tau(h) = \max |\tau_k(h)|$, prove that

$$|y(t_k) - y_k| \le e^{L(t_k - t_0)}|y(t_0) - y_0| + \frac{e^{L(t_k-t_0)} - 1}{L}\tau(h).$$

(You may use the inequality $1 + x \le e^x$ for any x.)