Preliminary Examination in Numerical Analysis

Jan. 7, 2015

Instructions:

1. The examination is for 3 hours.
2. The examination consists of ten equally-weighted problems. The first five cover Matrix Theory and Numerical Linear Algebra and the last five cover Introductory Numerical Analysis
3. You may **omit one** problem (i.e. work nine out of the ten problems).

Problem 1. Show that a (real) orthogonal matrix that is also upper triangular must be diagonal. What can be said about the diagonal elements?

Problem 2. Let $x_1, x_2, \cdots, x_n$ be floating point numbers. Prove that

$$\frac{|fl(x_1 x_2 \cdots x_n) - x_1 x_2 \cdots x_n|}{|x_1 x_2 \cdots x_n|} \le (n-1)\epsilon + \mathcal{O}(\epsilon^2)$$

where ϵ is the machine precision.

Problem 3. Let $A, E \in R^{n\times n}$ be symmetric matrices. If A is positive definite and $\|E\|_2 < \|A^{-1}\|_2^{-1}$, prove that $A+E$ is positive definite.
(Hint: prove first that $I+F$ is positive definite if F is symmetric and $\|F\|_2 < 1$.)

Problem 4. Let $A \in R^{m\times n}$ and $b \in R^m$ $(m \ge n)$. Let $A = U\Sigma V^T$ be the singular value decomposition of A, where

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \in R^{m\times n}; \ \Sigma_1 = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{pmatrix}$$

with $\sigma_1 \ge \cdots \ge \sigma_k \ge 0$ is $k \times k$. Determine with proofs when the least squares problem $\min_{x\in R^n} \|Ax - b\|_2$ has exactly one solution and when it has infinitely many solutions. Write down the solution or the solution set.

Problem 5. Consider the following subspace iterations:

Algorithm: Input A and an orthogonal $Q_0 \in R^{n\times p}$
For $i = 0, 1, 2, \cdots, m-1$
$U_i = AQ_i$
$U_i = Q_{i+1}R_{i+1}$ (QR-factorization)
End

After k iterations, prove that the columns of Q_k form an orthonormal basis that would be produced by applying the Gram-Schmidt orthogonalization process to the columns of $A^k Q_0$.

Problem 6. Consider the modified Newton iteration for finding the roots of a function $f : \mathbb{R} \to \mathbb{R}$,

$$x_{k+1} = x_k - \alpha\frac{f(x_k)}{f'(x_k)}.$$

Assuming that $f \in C^1[a,b]$, $f'(x) > 0$ for $x \in [a,b]$ and that f has only one root $x_* \in [a,b]$, determine a condition on α such that $x_k \to x_*$ for any $x_0 \in [a,b]$. What is the asymptotic rate of convergence? Is quadratic convergence possible?

Problem 7. Let $x_0, \ldots, x_n$ be distinct points with increasing values in the interval $[a,b]$ and let $\omega(x) = \prod_{i=0}^{n}(x - x_i)$.

a) Show that $\omega^{(n+1)}(x) = (n+1)!$.

b) Suppose f is a function satisfying $f(x_i) = 0$ for $i = 0, \ldots, n$ and suppose $f^{(n-1)}$ is continuous in $[a,b]$ and $f^{(n)}$ exists in (a,b). Show that there exists a ξ in (a,b) with $f^{(n)}(\xi) = 0$.

c) Let f be any function such that $f^{(n)}$ is continuous in $[a,b]$ and $f^{(n+1)}$ exists in (a,b). Let p be the polynomial of degree at most n satisfying $p(x_i) = f(x_i)$ for $i = 0,\dots,n$. Show that for each x in $[a,b]$ there exists a ξ in (a,b) with

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega(x).$$

(Hint: Let x be different from any of the x_i's and consider

$$g(t) = f(t) - p(t) - \frac{f(x) - p(x)}{\omega(x)}\omega(t).)$$

Problem 8. Assuming that $f \in C^2[a,b]$, show that the error for the midpoint rule is

$$E = \int_a^b f(x)dx - (b-a)f\left(\frac{a+b}{2}\right) = f''(\eta)\frac{(b-a)^3}{24},$$

where $\eta \in [a,b]$.

Problem 9. Let m and k be a positive integers with $k \leq m+1$ and let ω be a polynomial of degree $m+1$ having distinct real roots $x_0,\dots,x_m$ satisfying

$$\int_a^b \omega(t)p(t)\,dt = 0$$

for all polynomials p of degree up to $k-1$, where $a < b$. Show that there exists real numbers $\lambda_0,\dots,\lambda_m$ such that

$$\int_a^b p(t)\,dt = \sum_{j=0}^{m} \lambda_j p(x_j)$$

holds for all polynomials of degree up to $m+k$. (Hint: The case $k = m+1$ is Gaussian quadrature.)

Problem 10. Recall that the truncation error for the trapezoid rule,

$$y_{n+1} = y_n + \frac{h}{2}(f(x_{n+1}, y_{n+1}) + f(x_n, y_n)),$$

for solving the initial value problem $y' = f(x,y)$ with $y(0) = y_0$ is

$$\tau_n(h) = -\frac{h^2}{12}y'''(\xi_n)$$

for some $\xi_n \in [x_n, x_{n+1}]$, where y is the solution of the initial value problem. Suppose that f satisfies a Lipschitz condition

$$|f(x,u) - f(x,v)| \leq L|u - v|,$$

where L is the Lipschitz constant, and that $|y'''(x)| \leq M$ for some positive constant M. Show that the global error $e_n = y(x_n) - y_n$ satisfies the inequality

$$|e_{n+1}| \leq |e_n| + \frac{hL}{2}(|e_{n+1}| + |e_n|) + \frac{h^3}{12}M.$$

Assuming $e_0 = 0$, deduce for a constant step size $h > 0$ satisfying $hL < 2$, that

$$|e_n| \leq \frac{h^2 M}{12L}\left[\left(\frac{1+\frac{1}{2}hL}{1-\frac{1}{2}hL}\right)^n - 1\right].$$

Hence the global error is $O(h^2)$.