Preliminary Examination in Numerical Analysis

January 10, 2017

Instructions:

1. The examination is for 3 hours.
2. The examination consists of ten equally-weighted problems. The first five cover Matrix Theory and Numerical Linear Algebra and the last five cover Introductory Numerical Analysis
3. You may **omit one** problem (i.e. work nine out of the ten problems).

Problem 1. Let A and Q be two $n \times n$ real matrices and assume that Q is orthogonal. Prove that

$$\mathrm{fl}\,(AQ) = (A+E)Q, \quad \|E\|_2 \le n^3\epsilon\|A\|_2 + \mathcal{O}(\epsilon^2).$$

(You may use without proof that $\mathrm{fl}\,(\sum_{i=1}^n x_i y_i) = \sum_{i=1}^n x_i y_i(1+\delta_i)$ with $|\delta_i| \le n\epsilon + \mathcal{O}(\epsilon^2)$ and $\frac{1}{\sqrt{n}}\|A\|_1 \le \|A\|_2 \le \sqrt{n}\|A\|_1$.)

Problem 2. Let A be an invertible $n \times n$ matrix. Suppose $\hat{x}$ is an approximate solution to $Ax = b$ and let $r = b - A\hat{x}$. Show directly from the definitions that

$$\frac{1}{\kappa(A)}\frac{\|r\|}{\|b\|} \le \frac{\|x-\hat{x}\|}{\|x\|} \le \kappa(A)\frac{\|r\|}{\|b\|},$$

where $\kappa(A) = \|A\|\|A^{-1}\|$.

Problem 3. Let $A \in R^{n\times n}$ be a symmetric positive definite matrix.

1. Write down the Cholesky Algorithm for computing the Cholesky factorization $A = GG^T$
2. Prove that $|g_{ij}| \le \sqrt{a_{ii}}$ for any $1 \le j \le i \le n$, where $G = [g_{ij}]$.

Problem 4. Let $A \in R^{m\times n}$ with $r = \mathrm{rank}(A) < n$ and $b \in R^m$ $(m \ge n)$. Let $A = U\Sigma V^T$ be the SVD of A. Find the solution set to the least squares problem $\min_{x\in R^n} \|Ax - b\|_2$. For some $\alpha > \|A^\dagger b\|_2$, find a solution x with $\|x\|_2 = \alpha$.

Problem 5. Let x be a unit eigenvector of $A \in \mathbb{C}^{n\times n}$ corresponding to λ. Let H be the Householder reflection such that $Hx = e_1$, where $e_1 = [1, 0, \cdots, 0]^T$. Prove that

$$HAH^* = \begin{pmatrix} \lambda & T_{12} \\ 0 & T_{22} \end{pmatrix}$$

where $T_{12} \in \mathbb{C}^{1\times(n-1)}$ and $T_{22} \in \mathbb{C}^{(n-1)\times(n-1)}$.

Problem 6. Suppose $g \in C^1[a,b]$, and there exists a $\lambda \in (0,1)$ such that

$$|g(x) - g(y)| < \lambda|x-y| \quad \text{for} \quad x, y \in (a,b).$$

Show that there exists a unique $x_* \in [a,b]$ such that $x_* = g(x_*)$, and that the iteration $x_{i+1} = g(x_i)$ for any $x_0 \in (a,b)$ converges to x_* with rate of at most λ.

Problem 7. Let $x_0, \ldots, x_n$ be distinct numbers and let $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$ be given numbers. It is known that there exists a polynomial of degree at most $2n+1$ such that $p(x_i) = a_i$ and $p'(x_i) = b_i$ for all $i = 0, \ldots, n$. Show that p is unique.

Problem 8. Let $w(t)$ be a continuous positive function on the interval $(0,1)$ and let Π_n be vector space of all real polynomials of degree at most n, where $n \ge 1$. Define a norm on Π_n by

$$\|p\| = \sqrt{(p,p)}, \text{ where } (p,q) = \int_0^1 p(t)q(t)w(t)\,dt \text{ and } p, q \in \Pi_n.$$

Let p_n be an orthogonal polynomial of degree n so $(p, p_n) = 0$ for all $p \in \Pi_{n-1}$ and let k_n be the coefficient of t^n in $p_n(t)$. Find the best approximation in the norm to t^n by polynomials in Π_{n-1}.

Problem 9. The purpose of this problem is to solve (algebraically) for distinct numbers x_1 and x_2 and nonzero numbers c_1 and c_2 such that

$$\int_0^1 p(x)\,dx = c_1 p(x_1) + c_2 p(x_2) \tag{1}$$

for all polynomials p of degree at most 3.

a) Calculate $\alpha_k = \int_0^1 x^k\,dx$ for $k = 0, \ldots, 3$. Solve for numbers y_0 and y_1 such that the system of equations below are satisfied.

$$\begin{aligned} \alpha_2 + y_0\alpha_0 + y_1\alpha_1 &= 0, \\ \alpha_3 + y_0\alpha_1 + y_1\alpha_2 &= 0 \end{aligned}$$

b) Let $q(x) = y_0 + y_1 x + x^2$. Show that if (1) holds then

$$\begin{aligned} c_1 q(x_1) + c_2 q(x_2) &= 0, \\ c_1 x_1 q(x_1) + c_2 x_2 q(x_2) &= 0 \end{aligned}$$

c) Show that $q(x_1) = q(x_2) = 0$.

d) Find x_1, x_2, c_1, c_2.

Problem 10. What is the order of the following two step method for approximating the solution to $y' = f(x, y)$

$$y_{n+1} = 5y_n - 4y_{n-1} - 3hf(x_n, y_n).$$

Is the method stable?