Preliminary Examination in Numerical Analysis

Jan. 5, 2018

Instructions:

1. The examination is for 3 hours.
2. The examination consists of eight equally-weighted problems. The first four cover Matrix Theory and Numerical Linear Algebra and the last four cover Introductory Numerical Analysis.
3. Attempt **all** problems.

Problem 1. Show that the growth factor of the Gaussian elimination with partial pivoting applied to an $n \times n$ unreduced upper Hessenberg matrix $A = [a_{ij}]$ (i.e., $a_{ij} = 0$ for $i > j+1$) is bounded above by n.

Problem 2. If $A \in R^{n\times n}$ is an orthogonal matrix and $B \in R^{n\times k}$, prove that $(AB)^\dagger = B^\dagger A^{-1}$. Show that this is not true if A is invertible but not orthogonal by giving a counterexample.

Problem 3. To compute the Schur decomposition of A, why do we usually first reduce A to an upper Hessenberg matrix H and then apply the QR Algorithm to H rather than applying the QR algorithm directly to A? Explain carefully with precise statements but no proof is needed.

Problem 4. Let $T = [t_{ij}]$ be a symmetric tridiagonal matrix and let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of T. Prove that

$$\min_i |\lambda_i - t_{nn}| \leq |t_{n-1,n}|.$$

Problem 5. Consider $f \in C^2[a,b]$ with $\alpha \in [a,b]$ a simple root of f. Show that there exists an $\epsilon > 0$ such that if $x_0 \in [\alpha - \epsilon, \alpha + \epsilon]$ then the iterates of Newton's method applied to f, x_i, will converge quadratically to α.

Problem 6. Let $m \geq 2$ and let $x_k = \cos(k\pi/m)$ for $0 \leq k \leq m$. Define $U_{m-1}(x) = (x - x_1)\cdots(x - x_{m-1})$. It is known that

$$\int_{-1}^{1} q(x)U_{m-1}(x)\sqrt{1-x^2}\, dx = 0$$

for all polynomials $q(x)$ of degree at most $m-2$. Use this fact to show that there exist constants $A_0, \ldots, A_m$ such that

$$\int_{-1}^{1} \frac{p(x)}{\sqrt{1-x^2}}\, dx = \sum_{k=0}^{m} A_k p(x_k).$$

for all polynomials p of degree $2m-1$.

Problem 7. Let $p_0, \ldots, p_m$ be polynomials defined recursively by

$$p_0(x) = 1, \quad p_1(x) = a_0 x + b_0,$$

$$p_{n+1}(x) = (a_n x + b_n)p_n(x) - p_{n-1}(x), \quad 1 \leq n \leq m.$$

Show that

$$\sum_{n=0}^{m} a_n p_n(x) p_n(y) = \frac{p_m(x)p_{m+1}(y) - p_{m+1}(x)p_m(y)}{y - x}$$

for all integers $m \geq 0$. Hint: Multiply the recursion relation by $p_n(y)$, then switch x and y and subtract.

Problem 8. Show that the local truncation error for the Trapezoid Rule

$$y_{i+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_{i+1})\right)$$

for approximating the solution to $y' = f(x,y)$ on the interval $[x_i, x_{i+1}]$ is $-\frac{1}{12}(x_{i+1} - x_i)^2 y'''(\xi)$, where $\xi \in [x_i, x_{i+1}]$, provided that $y \in C^3(x_i, x_{i+1})$.