Preliminary Examination in Numerical Analysis

Jan. 19, 2021

Instructions:

1. The examination is for 3 hours.
2. The examination consists of eight equally-weighted problems.
3. Attempt all problems.

Problem 1. Suppose that $A \in \mathbb{C}^{n\times n}$ and $S = \sum_{k=0}^{\infty} A^k$. Show the matrix S is invertible if and only if the spectral radius of A is strictly smaller than one. Specify the inverse matrix of S when it exists. **Hint:** Show that $\lim_{n\to\infty} A^n = 0$. Use this limit and the partial sum $S_n = I + A + \ldots + A^n$ to obtain useful relationships between S and A.

Problem 2. Let A be a Hermitian positive definite matrix with the form $A = \begin{bmatrix} a_{11} & \omega^* \\ \omega & M \end{bmatrix}$. Assume the first step of the Cholesky factorization yields

$$A = \begin{bmatrix} \sqrt{a_{11}} & 0 \\ \omega/a_{11} & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & M - \omega\omega^*/a_{11} \end{bmatrix} \begin{bmatrix} \sqrt{a_{11}} & \omega^*/\sqrt{a_{11}} \\ 0 & I \end{bmatrix}.$$

Show that both M and $M - \omega\omega^*/a_{11}$ are Hermitian positive definite matrices.

Problem 3. Let $\{\mathbf{x}_1, \ldots \mathbf{x}_n\}$ be an orthonormal set of eigenvectors of an $n \times n$ matrix $A + \lambda I$ with associated eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. Express the solution of the equation $A\mathbf{x} = \mathbf{b}$ in terms of the $\mathbf{x}_i$'s and λ_i's if $\lambda_i - \lambda \neq 0$ for all $1 \leq i \leq n$.

Problem 4. Let $A \in \mathbb{R}^{m\times n}$ with $m \geq n$ have rank n, and $\mathbf{b} \in \mathbb{R}^m$.

(a) Show that the function $f(\mathbf{x}) = \|A\mathbf{x} - \mathbf{b}\|_2^2 + \lambda\|\mathbf{x}\|_2^2$ has a *unique* minimizer for any $\lambda > 0$. **Hint:** Derive the normal equation.

(b) Use the SVD of A to find the solution to the problem in Part (a). If $\lambda \to \infty$, what happens to the solution?

Problem 5. Let $A \in \mathbb{R}^{n\times n}$ and $\mathbf{x} \in \mathbb{R}^n$. Show that, barring overflow and underflow,

$$\mathrm{fl}(\mathbf{x}^T(A\mathbf{x})) = \mathbf{x}^T A\mathbf{x} + f \text{ with } |f| \leq |\mathbf{x}|^T|A||\mathbf{x}| \cdot 2n\epsilon + \mathcal{O}(\epsilon^2),$$

where $\mathrm{fl}(e)$ denotes the computation results of an expression e in a floating point arithmetic, ϵ is the machine epsilon, and $|\cdot|$ denotes entrywise absolute value.

Problem 6. Consider the data $f(0) = 0$, $f(1) = 1$, $f(2) = 0$, $f(3) = 1$, and $f(4) = 0$. Find both Lagrange's and Newton's interpolating polynomials that interpolate the function values of f.

Problem 7.

(a) Find constants c_0, c_1, c_2, c_3 such that the quadrature formula

$$\int_0^1 f(x)dx = c_0 f(0) + c_1 f(1) + c_2 f'(0) + c_3 f'(1)$$

has the highest possible *degree of precision* (i.e., the highest degree of polynomials for which this formula is exact). What is the highest degree of precision?

(b) Extend the quadrature formula in Part (a) to $\int_a^b f(t)dt$.

Problem 8. Consider the initial value problem

$$y'(t) = \cos(y(t)), \quad y(0) = y_0.$$

(a) Describe the implicit Euler's method to solve the problem. Find the region of absolute stability.

(b) Determine the order of truncation error and the corresponding principal error function.