Preliminary Examination in Numerical Analysis

January 4, 2023

Instructions:

1. The examination is for 3 hours.
2. The examination consists of eight equally-weighted problems.
3. Attempt all problems.

Problem 1. Let A and Q be two $n \times n$ real matrices and assume that Q is orthogonal. Prove that

$$fl\,(QA) = Q(A+E), \qquad \text{where} \qquad \|E\|_2 \le n^{5/2}\epsilon\|A\|_2 + \mathcal{O}(\epsilon^2)$$

where ϵ is the machine precision. (Hint: You may use without proof that $fl\,(\sum_{i=1}^n x_i y_i) = \sum_{i=1}^n x_i y_i(1+\delta_i)$ with $|\delta_i| \le n\epsilon + \mathcal{O}(\epsilon^2)$ and $\frac{1}{\sqrt{n}}\|A\|_1 \le \|A\|_2 \le \sqrt{n}\|A\|_1$.)

Problem 2. For any $n \times n$ real matrix A, $A_S = (A + A^T)/2$ is symmetric and is called the symmetric part of A. Prove that if A_S is positive definite, then A is nonsingular. Further prove that A has an LU factorization without using pivoting.

Problem 3. Let A, B and C be real matrices with dimensions such that the product $A^T C B^T$ is well defined. Let $\mathcal{X}$ be the set of all matrices minimizing $\|AXB - C\|_F$. Find the solution of the problem $\min_{X\in\mathcal{X}} \|X\|_F$. (*Hint:* Use the SVDs of A and B.)

Problem 4. Let $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ be the eigenvalues of a symmetric positive definite matrix $A \in \mathbb{R}^{n\times n}$. Prove that

$$\lambda_n = \min_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2}$$

where the minimum is taken over $\mathbf{x} \in \mathbb{R}^n$.

Problem 5. Suppose the equation $f(x) = 0$ has a root α with multiplicity $m \ge 2$, and Newton's method converges to α. Show that this convergence is only linear. How would you modify the method to obtain quadratic convergence?

Problem 6. Let $x_0, x_1, \ldots, x_n$ be $n+1$ distinct numbers in [a,b].

a. Construct polynomials $L_i(x)$ of degree n, $i = 0, 1, \ldots, n$, such that

$$L_i(x_k) = \delta_{ik} = \begin{cases} 1, & i = k, \\ 0, & i \neq k, \end{cases}$$

for $k = 0, 1, \ldots, n$.

b. Construct polynomials $H_i(x)$ of degree $2n$, $i = 0, 1, \ldots, n$, such that $H_i(x_k) = \delta_{ik}$, $k = 0, 1, \ldots, n$, and $H_i'(x_j) = 0$, $j = 0, 1, \ldots, n, j \neq i$.

c. Construct polynomials $K_i(x)$ of degree $2n$, $i = 0, 1, \ldots, n$, such that $K_i(x_k) = \delta_{ik}$, $k = 0, 1, \ldots, n$, and $K_i'(x_j) = 0$, $j = 1, \ldots, n$.

Problem 7. Prove that if $s(x)$ is a cubic spline that interpolates the function $g(x) \in C^2[a, b]$ at the knots $a = x_1 < x_2 < \cdots < x_n = b$ and satisfies the clamped conditions, i.e., $s'(a) = g'(a)$ and $s'(b) = g'(b)$, then

$$\int_a^b \left[g''(x)\right]^2 dx \geq \int_a^b \left[s''(x)\right]^2 dx$$

Problem 8. Consider the initial value problem $y'(t) = f(t, y)$, $a \leq t \leq b$, and $y(a) = \alpha$.

a. Let $a = t_0 < t_1 < t_2 < \cdots < t_N = b$ be a uniform grid on $[a, b]$ with grid size $h = (b - a)/N$. Show that

$$y'(t_i) = \frac{-y(t_{i+2}) + 4y(t_{i+1}) - 3y(t_i)}{2h} + \frac{h^2}{3} y^{(3)}(\xi_i),$$

for some ξ_i in (t_i, t_{i+2}), for $i = 0, \ldots, N - 2$.

b. Analyze the consistency, stability, and convergence of the following multi-step method

$$y_{i+2} = 4y_{i+1} - 3y_i - 2hf(t_i, y_i).$$

for the numerical solution of $y'(t) = f(t, y)$ with $y_0 = y(t_0)$ and $y_1 = y(t_1)$.