Preliminary Examination in Numerical Analysis

January 5, 2024

Instructions:

1. The examination is for 3 hours.
2. The examination consists of eight equally-weighted problems.
3. Attempt all problems.

Problem 1. Consider computing $Ax - b$ in a floating point arithmetic where $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, $x \in \mathbb{R}^n$, and $b = [b_i] \in \mathbb{R}^n$. Prove that

$$fl(Ax - b) = (A + \Delta)x - \hat{b}$$

for some $\Delta = [\delta_{ij}] \in \mathbb{R}^{n \times n}$ and $\hat{b} = [\hat{b}_i] \in \mathbb{R}^n$ where $|\delta_{ij}| \leq (n+1)\epsilon|a_{ij}| + \mathcal{O}(\epsilon^2)$, $|\hat{b}_i - b_i| \leq \epsilon|b_i|$, and ϵ is the machine roundoff unit (sometimes also called machine epsilon).
(You may use $fl(\sum_{i=1}^d x_i y_i) = \sum_{i=1}^d x_i y_i (1 + \delta_i)$, with $|\delta_i| \leq d\epsilon + \mathcal{O}(\epsilon^2)$.)

Problem 2. Let A be an $n \times n$ invertible real matrix and U and V be $n \times k$ (with $n \geq k$) real matrices. If $I - V^T A^{-1} U$ is invertible, prove that $B = \begin{pmatrix} A & U \\ V^T & I \end{pmatrix}$ is invertible and find its inverse B^{-1} in terms of A^{-1} and $(I - V^T A^{-1} U)^{-1}$.

Problem 3. There are three methods for solving the least squares problem $\min_{x \in \mathbb{R}^n} \|Ax - b\|_2$, where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $m \geq n$. Describe *one* of these methods and discuss how it compares with the other two methods in terms of stability and computational efficiency.

Problem 4. Let $A \in \mathbb{R}^{n \times n}$ and $H = \begin{bmatrix} I_n & A^T \\ A & I_n \end{bmatrix}$ be nonsingular where I_n is the n-by-n identity matrix. Find the condition number $\kappa_2(H)$ in terms of the singular values of A.

Problem 5. Suppose $f \in C^5[-1, 2]$ satisfies $|f^{(k)}(x)| \leq M_k$ for $k = 0, 1, \ldots, 5$, $x \in [-1, 2]$, and

$$f(-1) = 3, \quad f'(-1) = 1 \quad f(0) = 1, \quad f(2) = 5, \quad f'(2) = 3.$$

(a) Estimate $f(1)$ using Lagrange interpolation and express its maximum possible error.

(b) Estimate $f(1)$ using Hermite interpolation.

Problem 6. Assume $f \in C^2[0, 1]$. Show that

$$\int_0^1 f(x)\mathrm{d}x = f(0.5) + \frac{1}{24} f''(\xi), \quad 0 < \xi < 1.$$

Problem 7. Prove that the following Runge-Kutta method

$$K_1 = hf(t, y)$$
$$K_2 = hf\left(t + \frac{1}{2}h, y + \frac{1}{2}K_1\right)$$
$$K_3 = hf\left(t + \frac{3}{4}h, y + \frac{3}{4}K_2\right)$$
$$y(t+h) = y(t) + \frac{1}{9}\left(2K_1 + 3K_2 + 4K_3\right)$$

for the initial value problem $y' = f(t, y)$, where $f(t, y) = y + t$ and $y(0) = y_0$, has the local truncation error of $O(h^4)$.

Problem 8. Find all positive real values of α for which the linear two-step method

$$y_{n+2} = \alpha y_n + \frac{h}{3}\left[f\left(t_{n+2}, y_{n+2}\right) + 4f\left(t_{n+1}, y_{n+1}\right) + f\left(t_n, y_n\right)\right]$$

is zero-stable when solving the initial value problem $y'(t) = f(t, y)$, $y(0) = y_0$, and achieves the highest possible order of accuracy.