Preliminary Examination in Numerical Analysis

Jan 10, 2025

Instructions:

1. The examination is for 3 hours.
2. The examination consists of eight equally-weighted problems.
3. Attempt all problems.

Problem 1. Let A be a real $n \times n$ matrix.

(a) Prove that $\|A\|_2 = \|A^T\|_2$.

(b) Prove that $\|A^T A\|_2 = \|A\|_2^2$.

Problem 2. Let A be an invertible upper triangular matrix and let $X = [x_{ij}] \in \mathbb{R}^{n\times n}$ be an upper triangular matrix with all the diagonal entries being zero. Prove that $A - X$ is invertible and

$$\|(A - X)^{-1}\| \le \sum_{i=0}^{n-1} \|A^{-1}\|^{i+1}\|X\|^i.$$

Problem 3. Let A be an $m \times n$ matrix and $b \in \mathbb{R}^m$ with $m \ge n$. Suppose that A has full rank and $A = QR$ where Q is an $m \times n$ orthogonal matrix and R is an $n \times n$ upper triangular matrix with positive diagonal entries. Find the formula for x that minimizes $||\ Ax - b\ ||_2$ using $A = QR$ (you are not allowed to use the normal equation.).

Problem 4. Let $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n > 0$ be the singular values of $A \in \mathbb{R}^{m\times n}$. Prove that

$$\sigma_1 = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} \quad \text{and} \quad \sigma_n = \min_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}.$$

Problem 5. Construct the interpolating polynomial in Newton's form for the following table

x	-1	0	1	-2
$f(x)$	1	2	5	0

Problem 6. Describe the secant method for finding the root of a smooth function $f(x)$. Show that the method is locally superlinearly convergent and identify its order of convergence without proof.

Problem 7. Construct the following Newton-Cotes formula

$$\int_{-1}^{1} f(t)dt = a \cdot f(-1) + b \cdot f(0) + c \cdot f(1) + E(f).$$

Then, determine the degree of exactness of the formula and find an expression of the error function $E(f)$ in terms of appropriate derivatives of f.

Problem 8. Show that the following one-step method for the initial value problem $x'(t) = f(t, x)$ is A-stable:

$$x_{n+1} = x_n + \frac{1}{2}h\left(f(t_n, x_n) + f(t_{n+1}, x_{n+1})\right),$$

where h is the step size, i.e., $t_{n+1} = t_n + h$.