Preliminary Examination in Numerical Analysis

June 9, 2003

Instructions:

1. The examination is for 3 hours.
2. The examination consists of two parts:
   Part I: Matrix Theory and Numerical Linear Algebra
   Part II: Numerical Analysis
3. There are three problem sets in each part. **Work two out of the three problem sets for each part.**
4. All problems carry equal weights.

PART I - Matrix Theory and Numerical Linear Algebra
(Work two of the three problems in this part)

Problem 1.

(a) Assume that A and $A + \delta A$ are $n \times n$ invertible matrices and $\eta \equiv \kappa(A)\frac{\|\delta A\|}{\|A\|} < 1$. If $Ax = b$ and $(A + \delta A)\hat{x} = b + \delta b$, prove that

$$\frac{\|x - \hat{x}\|}{\|x\|} \leq \frac{\kappa(A)}{1-\eta}\left(\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|}\right)$$

where $\|\cdot\|$ is any matrix operator norm and $\kappa(A)$ is the condition number of A. (You may use without proof that $\|(I - X)^{-1}\| \leq (1 - \|X\|)^{-1}$ if $\|X\| < 1$.)

(b) Let U be an $n \times n$ upper triangular matrix and consider solving $Ux = b$ by backward substitution in a floating point arithmetic. Prove that the computed solution $\hat{x}$ satisfies $(U + \delta U)\hat{x} = b$ with $|\delta U| \leq n\epsilon|U| + O(\epsilon^2)$, where ϵ is the machine precision. (You may use $fl(\sum_{i=1}^{d} x_i y_i) = \sum_{i=1}^{d} x_i y_i(1 + \delta_i)$ with $|\delta_i| \leq d\epsilon + O(\epsilon^2)$.)

Problem 2.

(a) State the power method for computing the largest eigenvalue (in absolute value) of a matrix A. State and prove the convergence result for the case that A is symmetric.

(b) For $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $x_0 = \begin{pmatrix} 1 \\ a \end{pmatrix}$, find the sequence generated by the power method for A with x_0 as the initial vector. Discuss the convergence property of the sequence obtained. Is there any contradiction to the result in (a)? Explain.

(c) Write down the shifted QR algorithm for a matrix A. Prove that the matrices produced are all similar to the original matrix.

Problem 3.

(a) Describe an algorithm to compute a QR factorization of an $m \times n$ matrix A.

(b) Let $A \in R^{m \times n}$ and $b \in R^m$ $(m \geq n)$. Consider the least squares problem

$$\min_{x \in R^n} \|Ax - b\|_2. \tag{1}$$

1. Prove that if x satisfies $A^T Ax = A^T b$, then x is a solution to (??).
2. Derive a method to solve (??) via A's singular value decomposition

$$A = U\Sigma V^T = U\begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} V^T$$

where Σ_1 is $k \times k$.

Part II – Numerical Analysis

(Work two of the three problems in this part)

Problem 4. Suppose $g(x)$ is a C^1 function with a fixed point z, i. e. $g(z) = z$, and

$$|g'(z)| = \alpha \quad < 1$$

(a) Prove that a fixed point iteration will converge linearly to z from any point x_0 sufficiently close to z.

(b) What is the rate of convergence ?

(c) Perform one iteration of Newton's method on the system:

$$\begin{aligned} x_1^2 - 2x_1 - x_2 + 0.5 = 0 \\ x_1^2 + 4x_2^2 - 4 = 0 \end{aligned}$$

starting at point $(2, 0.25)$.

Problem 5. Outline the ideas and steps to derive a Gauss Formula

$$\int_{-1}^{1} f(x)dx = \sum_{i=0}^{n} A_i w_i f(x_i)$$

which is exact for all the polynomials of degree ≤ 3 on $[-1, 1]$.

(a) How many nodes (minimum number) are needed for Gauss Formula to be exact for all the polynomials of degree ≤ 3, i.e., what is n? and why?

(b) Use a theorem about orthogonal polynomials and the fact that 1, x, $x^2 - \frac{1}{3}$, $x^3 - \frac{3}{5}x$ are orthogonal on $[-1, 1]$ with weight function $w_i = 1$ to determine x_i.

(c) Use method of undetermined coefficient to find A_i and write the Gauss Formula.

Problem 6. Where $x'(t) = f(t, x)$, $x(0) = x_0$ and $f_n = f(t_n, x_n)$, the formula

$$x_{n+1} - (1-c)x_n - cx_{n-1} = \frac{h}{12}[(5-c)f_{n+1} + 8(1+c)f_n + (5c-1)f_{n-1}]$$

is known to be exact for all polynomials of degree m or less for all c.

(a) Determine c so that it will be exact for all polynomials of degree $m+1$. Find c and m.

(b) Using the c found in (a), is this method stable? strongly stable? is this method consistent? convergent?