Preliminary Examination in Numerical Analysis

June 3, 2004

Instructions:

- Examination is 3 hours: 9am to noon
- Part I: Numerical Linear Algebra
- Part II: Numerical Analysis
- Each part has 3 problems, and 6 problems total. You must do Problems 1 and 5, but have a choice of solving only 3 of the remaining 4 problems. Mark the problems that you wish to be graded in the table below (5 marks total).
- For problems with multiple parts, you may get full credit for a particular part using results of previous parts even though you fail to finish these previous parts

Problem No.	to be graded?	grade
1	$\surd$	
2		
3		
4		
5	$\surd$	
6		
Total		

Part I. Numerical Linear Algebra

Problem 1. **Eigenvalue Computation.** Let $A \in \mathbb{R}^{n\times n}$, the set of $n \times n$ real matrices.

1. Define the householder transformation $Q \in \mathbb{R}^{n\times n}$, and show how to transform a vector $x \in \mathbb{R}^n$ to αe_1 by a householder transformation, where $\alpha \in \mathbb{R}$ and e_1 is the first column of I_n, the $n \times n$ identity matrix.

2. Outline a way to transform A to an upper Hessenberg matrix H by a similarity transformation. Do A and H have the same eigenvalues? How do their eigenvectors related?

3. Basic QR iteration: $H_1 = H$; $H_k = Q_k R_k$, $H_{k+1} = R_k Q_k$ for $k = 1, 2, \ldots$. But this basic QR iteration is usually too slow to have much practical value. Shifts σ_k must be introduced to speed up convergence. Describe the current practice of the use of shifts in QR iteration for *real* matrix A. What do you expect R_k converging to?

4. Obviously we may apply QR iteration directly to A, instead of transforming A to H first. Why not?

Problem 2. Let $A \in \mathbb{R}^{(n+1)\times n}$ be in the "upper Hessenberg" form, i.e., the submatrix of A's first n rows is upper Hessenberg and A's last row is $a_{n+1,n}e_n^T$. Assume $a_{i+1,i} \neq 0$ for $1 \leq i \leq n$. Outline a method to solve

$$\min_x \|Ax - b\|_2.$$

Does it have a unique solution?

Problem 3. Let $A \in \mathbb{R}^{n\times n}$ be nonsingular.

1. Let x and $\widetilde{x} = x + \delta x$ be the solutions of $Ax = b$ and $A\widetilde{x} = b + \delta b$, respectively. Show
$$\frac{\|\delta x\|_1}{\|x\|_1} \leq \kappa_1(A)\frac{\|\delta b\|_1}{\|b\|_1},$$
where $\kappa_1(A) = \|A\|_1\|A^{-1}\|_1$.

2. Let $u, v \in \mathbb{R}^n$. If you have subroutine to solve $Ax = b$ for any given b, how can you solve $(A + uv^T)y = c$ by calling the subroutine finite many times?

Part II. Numerical Analysis

Problem 4.

1. Denote the intervals that arise in the bisection method by $[a_0, b_0]$, $[a_1, b_1]$, $\ldots$, $[a_n, b_n]$. Let the midpoint of each interval be $c_n = (a_n + b_n)/2$. Show that
$$|c_n - c_{n+1}| = 2^{-n-2}(b_0 - a_0).$$

2. Suppose the method of functional iteration is applied to the function $F(x) = 2x + qx^2$, $\frac{1}{2} \leq q \leq 1$. For which interval does this iteration $x_{n+1} = F(x_n)$ converge ? Show why.

3. What is the order of convergence for $x_{n+1} = F(x_n)$ above?

Problem 5.

1. Find the Chebyshev polynomial interpolation P_2 that approximates $f(x) = xe^{2x}$ on $[-1, 1]$ and find the error bound for $|f(x) - P_2(x)|$.

2. Find the best approximation to $f(x) = e^x + x$ by a polynomial $g(x) = c_1 x + c_0$ on the interval $[-1, 1]$ using the norm:
$$\|f\| = \left[\int_{-1}^{1} [f(x)]^2 dx\right]^{1/2}.$$

Problem 6.

1. Establish a formula of the following form with highest accuracy,
$$f'(x) \sim \frac{1}{2h}[af(x - 2h) + bf(x - h) + cf(x)].$$
What is a, b, c and the error ?

2. Using the method of undetermined coefficients, determine A and B in the following Adams-Bashforth formula:
$$x_{n+1} = x_n + h[Af_n + Bf_{n-1}].$$

3. Show that the following multistep method
$$x_n - x_{n-2} = h(f_n - 3f_{n-1} + 4f_{n-2})$$
is convergent.