Preliminary Examination in Numerical Analysis

June 3 2005

Instructions:

- Examination is 3 hours: 9am to noon
- Part I: Numerical Linear Algebra
- Part II: Numerical Analysis
- Each part has 3 problems, and 6 problems total. You must do Problems 2 and 4, but have a choice of solving only 3 of the remaining 4 problems. Mark the problems that you wish to be graded in the table below (5 marks total).
- For problems with multiple parts, you may get full credit for a particular part using results of previous parts even though you fail to finish these previous parts

Problem No.	to be graded?	grade
1		
2	$\surd$	
3		
4	$\surd$	
5		
6		
Total		

Part I. Numerical Linear Algebra

Problem 1. Let $A \in \mathbb{R}^{n\times n}$.

1. Prove that if $\|A\| < 1$, then $A^m \to 0$ as $m \to +\infty$, where $\|\cdot\|$ is a consistent matrix norm, i.e., $\|XY\| \le \|X\|\,\|Y\|$.
2. The spectral radius $\rho(A)$ is defined as $\max|\lambda|$ among all A's eigenvalues. Prove that $A^m \to 0$ as $m \to +\infty$ if and only if $\rho(A) < 1$.
3. Suppose A is Hermitian. Prove that $A^m \to 0$ as $m \to +\infty$ if and only if $\|A\|_2 < 1$.

Problem 2. Let $A \in \mathbb{R}^{n\times n}$ be dense.

1. State Schur eigendecomposition theorem.
2. State *real* Schur eigendecomposition theorem.
3. Outline a practical QR algorithm for computing the real Schur eigen-decomposition of A, explain and make sure that your algorithm costs $O(n^3)$.

Problem 3. Let $L \in \mathbb{R}^{n\times n}$ be lower triangular and consider linear system $Lx = b$.

1. Devise an algorithm to numerically solve $Lx = b$ for x.
2. Present a backward error analysis of your algorithm, i.e., show that the numerical solution $\widetilde{x}$ satisfies exactly $(L + E)\widetilde{x} = b + f$ for some error matrix E and error vector f. Give bounds on E and f.

Part II. Numerical Analysis

Problem 4.

1. (a) Prove that Newton's method converge quadratically (to a simple root if started near the zero). (b) What is the order of convergence if r is a double root of the function f, i.e, $f(r) = f'(r) = 0 \neq f''(r)$? Prove your answer.

2. If the method of functional iteration applied to $F(x) = 2 + (x - 2)^4$, starting with $x = 2.5$, what is the order of convergence ? Find the range of starting values for which this functional iteration converges. (Note 2 is a fixed point).

Problem 5.

1. (a) When deriving the following Gaussian quadrature, find the polynomial $q(x)$ which defines the nodes x_i.

$$\int_{-1}^{1} f(x)x^2 dx = \sum_{i=0}^{1} A_i f(x_i).$$

   (b) Prove that all the coefficients (A_i) in a Gaussian quadrature formula are all positive.

2. Find a formula

$$\int_{0}^{1} f(x)dx \sim af(0) + bf(\frac{1}{2}) + cf(1)$$

   that is exact when f is a polynomial of degree 2. What is this formula and why is this exact for polynomial of degree 2 ?

Problem 6.

1. Determine a, b, c and d, so that the cubic spline have minimum $\int_{-1}^{1} [S''(x)]^2 dx$:

$$S(x) = \begin{cases} 3 - 2x^2, & x \in [-1, 0], \\ a + bx + cx^2 + dx^3, & x \in [0, 1] \end{cases}$$

2. Show that the following multistep method is convergent.

$$x_n - x_{n-2} = h(\frac{7}{3}f_{n-1} - \frac{2}{3}f_{n-2} + \frac{1}{3}f_{n-3}).$$