# Preliminary Examination in Numerical Analysis

June 2, 2006

**Instructions:**

1. The examination is for 3 hours.
2. The examination consists of two parts:
   Part I: Matrix Theory and Numerical Linear Algebra
   Part II: Introductory Numerical Analysis
3. There are three problem sets in each part. **Work two out of the three problem sets for each part.**
4. All problem sets carry equal weights.

PART I - Matrix Theory and Numerical Linear Algebra
**(Work two of the three problem sets in this part)**

**Problem 1**.

1. Assume that $A$ and $A + \delta A$ are $n \times n$ invertible matrices and $\eta := \kappa(A)\frac{\|\delta A\|}{\|A\|} < 1$. Prove that
$$\frac{\|(A+\delta A)^{-1} - A^{-1}\|}{\|A^{-1}\|} \leq \frac{\kappa(A)\frac{\|\delta A\|}{\|A\|}}{1-\eta}$$
where $\|\cdot\|$ is any matrix operator norm and $\kappa(A)$ is the condition number of $A$.

2. Let $A \in R^{n\times n}$ be symmetric positive definite. Prove by induction on $n$ that $A$ has a Cholesky factorization, namely there exists a lower triangular matrix $G$ such that $A = GG^T$.

3. Write down the algorithm for computing the Cholesky factorization. Use the error analysis of the $LU$ factorization to explain briefly why the Cholesky algorithm is stable without pivoting.

**Problem 2**. Let $A$ be an $m \times n$ matrix and $m > n$.

1. Describe the Gram-Schmidt process to orthogonalize the columns of $A$. Prove that the process produces an orthonormal basis if the columns of $A$ are linearly independent. Formulate the process into the matrix factorization $A = QR$, and what is $Q$ and what is $R$?

2. Describe an algorithm for computing the $QR$ factorization of $A$ by the Householder transformations. How does this algorithm compare with the one based on the Gram-Schmidt process?

**Problem 3**.

1. State the power method for computing the largest eigenvalue (in absolute value) of a matrix $A \in R^{n\times n}$. State and prove the convergence result for the case that $A$ is symmetric. Describe the shift-and-invert method to compute the eigenvalue closest to a given number $\mu$.

2. Let $\beta$ be an approximate eigenvalue and $x \in R^n$ with $\|x\|_2 = 1$ a corresponding approximate eigenvector of an $n \times n$ real symmetric matrix $A$. Prove
$$\min_i |\lambda_i - \beta| \leq \|Ax - \beta x\|_2.$$
where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of $A$.

## Part II – Numerical Analysis

**(Work two of the three problem sets in this part)**

**Problem 4**. Polynomial interpolation.

1. Prove that if $g$ interpolates the function $f$ at $x_0, x_1, \ldots, x_{n-1}$ and if $h$ interpolates $f$ at $x_1, x_2, \ldots, x_n$, then the function

$$g(x) + \frac{x_0 - x}{x_n - x_0}[g(x) - h(x)]$$

interpolates $f$ at $x_0, x_1, \ldots, x_{n-1}, x_n$.

2. Complete the following divided-difference table

| $x$ | $f(x)$ | $f[\ ,\ ]$ | $f[\ ,\ ,\ ]$ | $f[\ ,\ ,\ ,\ ]$ |
|---|---|---|---|---|
| 4 | 63 | | | |
| 2 | 11 | | | |
| 0 | 7 | | | |
| 3 | 28 | | | |

3. Obtain polynomials of degree 3 that interpolates the function values given in (2), using Newton's formula.

4. Obtain polynomials of degree 3 that interpolates the function values given in (2), using Lagrange's formula. (You may just write down the formula without simplifying it.)

**Problem 5**. Quadrature rules.

1. Given $(n+1)$ nodes $x_0, x_1, \ldots, x_n$ in $[a, b]$. Derive the Newton-Cotes formula for approximating $\int_a^b f(x)\, dx$. Write down the trapezoid rule (for which $n = 1$, $x_0 = a$, and $x_1 = b$).

2. Prove that the Newton-Cotes formula you just derived is exact for all polynomials of degree no higher than $n$.

3. Find the Gaussian quadrature rule

$$\int_{-1}^{1} f(x)\, dx \approx A_0 f(x_0) + A_1 f(x_1) + A_2 f(x_2).$$

What is the highest degree of polynomials for which this formula is exact?

4. Use the Gaussian quadrature rule you just found to find the Gaussian quadrature rule

$$\int_a^b g(t)\, dt \approx A_0 f(t_0) + A_1 f(t_1) + A_2 f(t_2).$$

Use it to approximate $\int_0^2 \frac{\sin t}{t}\, dt$.

**Problem 6**. Zeros of functions and splines.

1. What are the conditions that guarantee the bisection to work, i.e., to find a zero? Justify your answer.

2. What are the conditions that guarantee the Newton method to converge quadratically at the end? Justify your answer.

3. Given $(n+1)$ knots $t_0 < t_1 < \cdots < t_n$, what are the conditions to make a function $S(x)$ on $[t_0, t_n]$ a natural cubic spline.

4. Find the natural cubic spline that interpolates the table

| $x$ | $-1$ | $0$ | $1$ |
|---|---|---|---|
| $y$ | $1$ | $2$ | $-3$ |