Preliminary Examination in Numerical Analysis

June 1, 2007

Instructions:

1. The examination is for 3 hours.
2. The examination consists of two parts:
   Part I: Matrix Theory and Numerical Linear Algebra
   Part II: Introductory Numerical Analysis
3. There are three problem sets in each part. **Work two out of the three problem sets for each part.**
4. All problem sets carry equal weights.
5. Problems within each problem set are not necessarily related but they may be. You could use the result from one part in your solutions for other parts, even if you did not prove it.

PART I - Matrix Theory and Numerical Linear Algebra
(Work two of the three problem sets in this part)

Problem 1. Let $\mathrm{fl}\,(x)$ denote the computational result of an expression x in a floating point arithmetic and let ϵ be the machine roundoff unit. Let $|A|$ denote the matrix obtained from A by applying absolute values in each entry.

a) Let $A,\ B \in R^{n\times n}$. Prove that

$$\mathrm{fl}\,(AB) = AB + E, \quad |E| \le n\epsilon |A|\,|B| + \mathcal{O}(\epsilon^2).$$

You can use without proof that

$$\mathrm{fl}\,(\sum_{i=1}^{n} x_i y_i) = \sum_{i=1}^{n} x_i y_i (1+\delta_i)$$

with $|\delta_i| \le n\epsilon + \mathcal{O}(\epsilon^2)$.

b) Let A and X be two $n \times n$ matrices, and assume that X is nonsingular. Show that

$$\mathrm{fl}\,(AX) = (A + \hat{E})X, \quad \|\hat{E}\|_1 \le n\epsilon\kappa_1(X)\|A\|_1,$$

where $\kappa_1(X) = \|X\|_1 \|X^{-1}\|_1$.

c) Let A and Q be two $n \times n$ matrices, and assume that Q is orthogonal. Show that

$$\mathrm{fl}\,(AQ) = (A + \hat{E})Q, \quad \|\hat{E}\|_2 \le n^3\epsilon \|A\|_2 + \mathcal{O}(\epsilon^2).$$

(You may use without proof that $\|A\|_2 \le \sqrt{n}\|A\|_1$ and $\|A\|_1 \le \sqrt{n}\|A\|_2$.)

Problem 2.

a) Let A be an $n \times n$ nonsingular matrix and $\hat{x}$ be an approximate solution to $Ax = b$. If $r = A\hat{x} - b$, prove that

$$\frac{\|\hat{x} - x\|}{\|x\|} \le \kappa(A)\frac{\|r\|}{\|b\|},$$

where $\kappa(A) = \|A\|\|A^{-1}\|$.

b) Describe the algorithm for computing the Cholesky factorization of a symmetric positive definite matrix. Explain its backward stability.

c) Given two vectors $x, y \in R^n$ with $\|x\|_2 = \|y\|_2$, use the Householder reflection to find an orthogonal matrix P such that $Px = y$.

Problem 3.

a) Describe an efficient algorithm to reduce the following matrix

$$A = \begin{pmatrix} \mathtt{x} & \mathtt{x} & \mathtt{x} & \mathtt{x} & \mathtt{x} & \mathtt{x} \\ \mathtt{x} & \mathtt{x} & \mathtt{x} & \mathtt{x} & \mathtt{x} & \mathtt{x} \\ \mathtt{x} & \mathtt{x} & \mathtt{x} & \mathtt{x} & \mathtt{x} & \mathtt{x} \\ & & \mathtt{x} & \mathtt{x} & \mathtt{x} & \mathtt{x} \\ & & & \mathtt{x} & \mathtt{x} & \mathtt{x} \\ & & & & \mathtt{x} & \mathtt{x} \end{pmatrix}$$

to the upper Hessenberg form by an orthogonal similarity transformation. (A sketch of a procedure is sufficient.)

b) If $H = [h_{ij}]$ is an unreduced upper Hessenberg matrix (i.e. $h_{i+1,i} \neq 0$ for $1 \leq i \leq n-1$) and $H = QR$ with $R = [r_{ij}]$ is its QR factorization, then $r_{ii} \neq 0$ for $1 \leq i \leq n-1$. Prove this statement for the case of 3×3 matrices.

c) If λ is an eigenvalue of an unreduced upper Hessenberg matrix H, prove that one step of the single shifted QR iteration with λ as the shift produces a Hessenberg matrix with the last row equal to $[0, \cdots, 0, \lambda]$.
(Hint: use part (b) and the fact that $H - \lambda I$ is singular.)

Part II – Numerical Analysis

(Work two of the three problem sets in this part)

Problem 4.

1. Suppose that r is a root of order 3 of the function f, i.e. $f(r) = f'(r) = f''(r) = 0 \neq f'''(r)$. What is the order of convergence for the following method ? (Show work.)

$$x_{n+1} = x_n - 3f(x_n)/f'(x_n)$$

2. (a) Suppose the method of functional iteration is applied to the function $F(x) = qx^2 + 3$, $\frac{1}{2} \leq q \leq 1$. On what interval can it be guaranteed that the method of iteration using F will converge to a fixed point, starting from any point in that interval ?

   (b) If f' is continuous and positive on $[a, b]$, and if $f(a)f(b) < 0$, then f has exactly one zero in (a, b). For what value of λ can the zero be obtained by applying the method of functional iteration with $F(x) = x + \lambda f(x)$? (Show work.)

Problem 5.

1. (a) Find a formula

$$\int_0^1 f(x)dx \sim af(\frac{1}{4}) + bf(\frac{1}{2}) + cf(\frac{3}{4})$$

   that is exact when f is a polynomial of degree 2.

   (b) What is the error associated with this method ? (You must show work.)

2. Find a nonzero polynomial $q(x)$ of degree 3 that can be used to find the nodes x_i in the Gaussian quadrature formula

$$\int_{-1}^1 f(x)w(x)dx \approx \sum_{i=0}^{2} A_i f(x_i).$$

   This approximation should be exact for all $f \in \Pi_5$ (the space of polynomials of degree ≤ 5) with the weight function $w(x) = x^2$. Clearly state the condition needed for $q(x)$.

Problem 6.

1. Find the best approximation to $f(x) = x\sin(\pi x)$ by a polynomial $g(x) = Ax + B$ on the interval $[-1, 1]$ using the norm,

$$\|f\| = \{\int_{-1}^1 [f(x)]^2 dx\}^{1/2}.$$

2. (a) Consider numerically solving the initial-value problem

$$\begin{cases} x' = f(t, x) \\ x(t_o) = x_o. \end{cases}$$

   Derive a multistep formula of the form,

$$x_{n+1} = x_n + h(\alpha f_{n+1} - \beta f_n + \gamma f_{n-1}).$$

   (b) Discuss the convergence of above method.