Preliminary Examination in Numerical Analysis

June 3, 2009

Instructions:

1. The examination is for 3 hours.
2. The examination consists of two parts:
   Part I: Matrix Theory and Numerical Linear Algebra
   Part II: Introductory Numerical Analysis
3. There are three problem sets in each part. **Work two out of the three problem sets for each part.**
4. All problem sets carry equal weights (but not necessarily the various problems within each problem set).
5. Problems within each problem set are not necessarily related but they may be. You could use the result from one part in your solutions for other parts, even if you did not prove it.

PART I - Matrix Theory and Numerical Linear Algebra

(Work two of the three problem sets in this part)

Problem 1.

1. Let A and E be two $n \times n$ matrices such that A is invertible and $\eta \equiv \kappa(A)\frac{\|E\|}{\|A\|} < 1$ where $\|\cdot\|$ is a matrix operator norm and $\kappa(A) = \|A\|\|A^{-1}\|$ is the condition number of A. Prove that $A + E$ is invertible and

$$\frac{\|(A+E)^{-1} - A^{-1}\|}{\|A^{-1}\|} \leq \frac{\kappa(A)\frac{\|E\|}{\|A\|}}{1-\eta}.$$

2. A matrix $A = [a_{ij}]$ is said to be *strictly column diagonally dominant* if $|a_{ii}| > \sum_{j \neq i} |a_{ji}|$ for all i.
   (a) Applying one step of the Gaussian elimination to A, show that the resulting matrix is still strictly column diagonally dominant.
   (b) Use induction to prove that the LU factorization for A exists.

Problem 2.

1. Let μ be an approximate eigenvalue of an $n \times n$ matrix A and x be an approximate eigenvector with $\|x\|_2 = 1$. If $r = Ax - \mu x$, show that there is a matrix E with $\|E\|_2 = \|r\|_2$ such that $A + E$ has eigenvalue μ and eigenvector x.

2. Describe a method to transform an $n \times n$ matrix A to an upper Hessenberg matrix H by an orthogonal similarity transformation through a sequence of Householder reflections.

3. To compute the Schur decomposition of A, why do we usually first reduce A to an upper Hessenberg matrix H and then apply the QR Algorithm to H rather than applying the QR algorithm directly to A? Explain carefully with precise statements but no proof is needed.

Problem 3. Let A be $m \times n$ and $m > n$.

1. Describe the Gram-Schmidt process to orthogonalize the columns of A. (A statement of the algorithm is sufficient.) Formulate the process into the matrix factorization $A = QR$, and what is Q and what is R?

2. Does Gram-Schmidt process always produce vectors orthogonal up to around the machine precision in a floating point arithmetic? Describe the Modified Gram-Schmidt (MGS) process. Does MGS always produce vectors orthogonal up to around the machine precision?

3. Given the QR factorization of $A = QR$, derive a method to solve the least squares problem $\min_{x \in R^n} \|b - Ax\|_2$, where $b \in R^m$.

Part II – Numerical Analysis
(Work two of the three problems in this part)

Problem 4.

(a) State the Contractive Mapping Theorem.

(b) Prove the Contractive Mapping Theorem.

(c) Generate the Gaussian quadrature formula $\int_{-1}^{1} f(x)\mathrm{d}x \approx \sum_{i=0}^{n} A_i f(x_i)$ having the smallest possible number of quadrature nodes x_i that is exact for all polynomials of degree 3 and less. Be sure to explain how the quadrature nodes x_i and the weights A_i are found.

Problem 5.

(a) In the following table, one of the most popular nonlinear root-finding methods (bisection, secant, or Newton) was applied to the function $f(x) = e^{2x} - 1$ using double-precision arithmetic in order to approximate the root $r = 0$.

i	0	1	2	3	4	5	6	7
x_i	1.204	0.7486	0.3605	0.1036	0.01004	1.000×10^{-4}	1.001×10^{-8}	1.095×10^{-16}

Based on the order of convergence observed in the table above for large enough values of i, state which of the methods (bisection, secant or Newton) was most likely used to generate the table.

(b) As a continuation of part (a), state and verify the conditions on f that guarantee theoretically that the order of convergence observed above must be achieved.

(c) Use the Method of Undetermined Coefficients to generate the third-order Adams-Bashform method of the form $x_{n+1} = x_n + h[Af_n + Bf_{n-1} + Cf_{n-2}]$.

Problem 6.

(a) Let $f(x) = \sin(\pi x/2)$. Use Hermite interpolation to find a polynomial p in Newton form which interpolates f and f' at $x_0 = -1$ and $x_1 = 1$, that is, such that $p(x_i) = f(x_i)$ and $p'(x_i) = f'(x_i)$, $i = 0, 1$.

(b) Find a bound for $\max_{-1 \le x \le 1} |f(x) - p(x)|$. (You may cite a theorem as part of your solution, but your final answer should be a number.)

(c) The general form of a linear multistep method is $a_k x_n + a_{k-1} x_{n-1} + ... + a_0 x_{n-k} = h[b_k f_n + b_{k-1} f_{n-1} + ... + b_0 f_{n-k}]$. Let $p(z) = \sum_{i=0}^{k} a_i z^i$, and $q(z) = \sum_{i=0}^{k} b_i z^i$. Show that if the above multistep method is consistent and $p(z) = q(z)$, then the method is also unstable.