Preliminary Examination in Numerical Analysis

June 3, 2010

Instructions:

1. The examination is for 3 hours.
2. The examination consists of two parts:
   Part I: Matrix Theory and Numerical Linear Algebra
   Part II: Introductory Numerical Analysis
3. There are three problem sets in each part. **Work two out of the three problem sets for each part.**
4. All problem sets carry equal weights.
5. Problems within each problem set are not necessarily related but they may be. You could use the result from one part in your solutions for other parts, even if you did not prove it.

PART I - Matrix Theory and Numerical Linear Algebra
(Work two of the three problem sets in this part)

Problem 1.

1. Assume that A and $A - \delta A$ are invertible and $\|A^{-1}\|\|\delta A\| < 1/2$. Prove
$$\frac{\|(A-\delta A)^{-1} - A^{-1}(A+\delta A)A^{-1}\|}{\|A^{-1}\|} \leq 2\|A^{-1}\|^2\|\delta A\|^2.$$
where $\|\cdot\|$ is a matrix operator norm.

2. Let U be an $n \times n$ upper triangular matrix and consider solving $Ux = b$ by backward substitution in a floating point arithmetic. Prove that the computed solution $\hat{x}$ satisfies $(U + \delta U)\hat{x} = b$ with $|\delta U| \leq n\epsilon|U| + O(\epsilon^2)$, where ϵ is the machine precision. (You may use $fl(\sum_{i=1}^{d} x_i y_i) = \sum_{i=1}^{d} x_i y_i (1+\delta_i)$ with $|\delta_i| \leq d\epsilon + O(\epsilon^2)$.)

3. Let $A = [a_{ij}]$ be an $n \times n$ symmetric positive definite matrix and let $A = GG^T$ be its Cholesky factorization, where $G = [g_{ij}]$ is lower triangular. Let $B = [b_{ij}] = |G||G|^T$. Prove that
$$|b_{ij}| \leq \max_i a_{ii}.$$

Problem 2.

1. Let $A \in \mathbb{R}^{(n+1)\times n}$ be in the upper Hessenberg form, i.e., $a_{ij} = 0$ for all $i > j+1$. Assume $a_{i+1,i} \neq 0$ for $1 \leq i \leq n$. Describe an efficient algorithm (with $O(n^2)$ operations) to solve
$$\min_x \|Ax - b\|_2.$$
Show that the solution is unique.

2. Let A be an $m \times n$ matrix and B be an $m \times k$ matrix. Prove that $X = A^+B$ minimizes $\|AX - B\|_F$ over all $n \times k$ matrices X.

3. Let $A \in R^{m\times n}$ and $b \in R^m$ $(m \geq n)$. Prove that x is a solution to the least squares problem $\min_x \|Ax - b\|_2$ if and only if x satisfies
$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

Problem 3.

1. Let β be an approximate eigenvalue and $x \in R^n$ with $\|x\|_2 = 1$ a corresponding approximate eigenvector of an $n \times n$ real symmetric matrix A. Prove that
$$\min_i |\lambda_i - \beta| \leq \|Ax - \beta x\|_2$$
where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of A.

2. Describe an algorithm to reduce a symmetric matrix to a tridiagonal matrix through a sequence of orthogonal similarity transformations.

3. Write down the (unshifted) QR algorithm for an $n \times n$ symmetric matrix A. If A is tridiagonal, show that the tridiagonal form is preserved by the QR algorithm.

Part II – Numerical Analysis
(Work two of the three problem sets in this part)

Problem 4. Assume that $f \in C^2(\mathbf{R})$, $f' > 0$, $f'' > 0$, and $f(r) = 0$.

a) Let $\{x_n\}$ denote the iterates of Newton's method for approximating r. Show that

$$r - x_{n+1} = -\frac{f''(\xi_n)}{2f'(x_n)}(r - x_n)^2 ,$$

where $\min(x_n, r) < \xi_n < \max(x_n, r)$.

b) Prove that, if Newton's method is started with any initial iterate $x_0 \in \mathbf{R}$, then $x_n \to r$.

c) Given below are parts of the convergence histories, $\ln 2 - x_n$, of the bisection method, the secant method and Newton's method for approximating the zero of $f(x) = e^x - 2$.

A	B	C
-1.31e+00	-9.31e+00	-2.00e-09
2.36e+00	-8.31e+00	-9.50e-10
1.44e+00	-7.31e+00	-4.25e-10
-3.50e+00	-6.31e+00	-1.63e-10
1.33e+00	-5.31e+00	-3.13e-11
1.22e+00	-4.31e+00	3.44e-11
-1.29e+00	-3.33e+00	1.56e-12
6.90e-01	-2.36e+00	-1.48e-11
3.75e-01	-1.46e+00	-6.64e-12
-1.55e-01	-6.91e-01	-2.54e-12
3.01e-02	-1.92e-01	-4.88e-13
2.29e-03	-1.73e-02	5.37e-13
-3.46e-05	-1.49e-04	2.44e-14
3.95e-08	-1.10e-08	-2.32e-13
6.83e-13	-1.11e-16	-1.04e-13

Match each of the columns A, B and C to each of the three root-finding methods. No justification is needed.

Problem 5.

a) Show that if p is a polynomial of degree three or less, then Simpson's rule is exact:

$$\int_a^b p(y)\,dy = \frac{b-a}{6}(p(a) + 4p(c) + p(b)), \text{ where } c = \frac{a+b}{2}.$$

b) Argue that, for general $g \in C^4[a,b]$, the error in approximating $\int_a^b g(y)\,dy$ via Simpson's rule is $\mathcal{O}((b-a)^5)$.

c) Consider the IVP $x' = f(t,x)$, $x(t_0) = x_0$, and the following multi-step method:

$$x_{n+1} = x_{n-1} + \frac{h}{3}(f(t_{n-1}, x_{n-1}) + 4f(t_n, x_n) + f(t_{n+1}, x_{n+1})).$$

Here, $t_j = t_0 + jh$ for some given $h > 0$. Assuming that x is sufficiently smooth, what is the order of the local truncation error for this method? Justify your answer.

Problem 6. Assume that $f \in C^\infty(\mathbb{R})$ and $|f^{(j)}(x)| \le 2^j$ for $j \ge 0$ and $x \in [-2, 2]$.

a) Provide the sharpest possible upper-bound on $|f'(1/2) - (f(1) - f(0))|$ based on the given information.

b) Consider the IVP $x' = f(x)$, $x(0) = 0$.

i) Argue that this problem has a unique solution for $t \in [0,1]$.

ii) Let $N \in \mathbb{N}$ and $h = 1/N$. The second-order Taylor iteration is $x_0 = 0$ and $x_{n+1} = x_n + hf(x_n)(1 + 0.5hf'(x_n))$ for $0 \le n < N$. Prove that $|x_n| \le n\frac{N+1}{N^2}$.

c) Suppose that

x	$f(x)$	$f'(x)$	$f''(x)$
0	1	0	-4
1	0		
2	1		

Give the Hermite interpolant of this data in its Newton form.