Preliminary Examination in Numerical Analysis

June 2011

Instructions:

1. The examination is for 3 hours.
2. The examination consists of two parts:
   Part I: Matrix Theory and Numerical Linear Algebra
   Part II: Introductory Numerical Analysis
3. There are three problem sets in each part. **Work two out of the three problem sets for each part.**
4. All problem sets carry equal weights.
5. Problems within each problem set are not necessarily related but they may be. You could use the result from one part in your solutions for other parts, even if you did not prove it.

PART I - Matrix Theory and Numerical Linear Algebra

(Work two of the three problem sets in this part)

Problem 1.

1. Let A be an $n \times n$ nonsingular matrix and let $\hat{x}$ be an approximate solution to $Ax = b$. If $r = A\hat{x} - b$, prove that
$$(A + E)\hat{x} = b$$
where E is a matrix satisfying $\|E\|_2 = \frac{\|r\|_2}{\|\hat{x}\|_2}$.

2. Let A be an $n \times n$ invertible matrix and U and V be $n \times k$ (with $n \geq k$) matrices. Prove that the following are equivalent:

   (a) $\begin{pmatrix} A & U \\ V^T & I \end{pmatrix}$ is invertible;

   (b) $I - V^T A^{-1} U$ is invertible;

   (c) $A - UV^T$ is invertible.

   Assuming that $T = I - V^T A^{-1} U$ is invertible, prove that
$$(A - UV^T)^{-1} = A^{-1} + A^{-1} U T^{-1} V^T A^{-1}.$$

Problem 2.

1. Let A be an $m \times n$ matrix with $\text{rank}(A) = n \leq m$ and $b \in R^n$. Let $A = U\Sigma V^T$ be the singular value decomposition of A, where
$$U \in R^{m\times n},\ \Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix}, \quad \text{and } V \in R^{n\times n}$$
with $\sigma_1 \geq \cdots \geq \sigma_n > 0$.

   (a) Determine the singular value decompositions of $(A^T A)^{-1} A^T$ and $A(A^T A)^{-1} A^T$.

   (b) If x is the solution to the least squares problem $\min \|Ax - b\|_2$ and $\hat{x}$ is the solution to a perturbed problem $\min \|Ax - \hat{b}\|_2$, prove that
$$\hat{x} - x = (A^T A)^{-1} A^T (\hat{b} - b)$$
   and then
$$\frac{\|\hat{x} - x\|_2}{\|x\|_2} \leq \kappa \frac{\|\hat{b} - b\|_2}{\|U^T b\|_2}$$
   where $\kappa = \sigma_1 / \sigma_n$.

2. Let $A = [a_{ij}] \in \mathbb{R}^{(n+1)\times n}$ be in the upper Hessenberg form, i.e., $a_{i,j} = 0$ for all $i > j+1$. Assume that $a_{i+1,i} \neq 0$ for $1 \leq i \leq n$. Describe an efficient method (with $O(n^2)$ operations) to solve
$$\min_x \|Ax - b\|_2.$$
Does the least squares problem have a unique solution? Explain.

Problem 3. Let $A \in \mathbb{R}^{n\times n}$.

1. State the power method;
2. Present a convergence analysis of the power method, assuming that A is diagonalizable with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ such that $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$;
3. Given a fixed number μ, describe how we can use the power method to compute the eigenvalue that is closest to μ?
4. For $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $x_0 = \begin{pmatrix} 1 \\ a \end{pmatrix}$, find the sequence generated by the power method for A with x_0 as the initial vector. Discuss the convergence property of the sequence obtained. Is there any contradiction to the convergence analysis? Explain.

PART 2 - Numerical Analysis

(Work two of the three problem sets in this part)

Problem 4.

1. Show that the error term for the Trapezoid Rule on the interval $[a, b]$ is $-\frac{1}{12}(b-a)^3 f''(\xi)$, where $\xi \in [a, b]$. More precisely, show that if f is sufficiently smooth, then
$$\int_a^b f(x)\mathrm{d}x = (b-a)\frac{f(a)+f(b)}{2} - \frac{1}{12}f''(\xi)(b-a)^3.$$

2. Suppose $f : [a, b] \to [a, b]$, $f'(x) < 1$ on $[a, b]$, and f' is continuous on $[a, b]$. Show that f has a fixed point $s \in [a, b]$.

3. Find an interpolating polynomial $p(x)$ in Newton form satisfying $p(1) = 1$, $p'(1) = 2$, $p''(1) = -1$, $p(2) = 3$.

Problem 5.

1. Suppose that $L = \phi(h) + \sum_{j=1}^{\infty} a_{3j} h^{3j}$. Define a Richardson extrapolation procedure for generating successively high-order approximations to L. Be sure to give an algorithm for generating terms of arbitrarily high order (that is, only generating the first few extrapolations is NOT sufficient to receive full credit).

2. Second-order Runge-Kutta methods may in general be derived by writing $x(t+h) = x + w_1 hf + w_2 hf(t + \alpha h, x + \beta hf) + O(h^3)$. Here we use the convention $f = f(t, x)$ and $x = x(t)$. Using a multivariate Taylor expansion of f about (t, x), find conditions on w_1, w_2, α, and β that ensure that the error term in this formula is in fact $O(h^3)$. Then give two different concrete examples of second-order Runge-Kutta methods.

Problem 6.

1. Use the method of undetermined coefficients to determine a linear multistep method of the form $x_{n+1} = x_n + h[Af_n + Bf_{n-2} + Cf_{n-4}]$, where $f_j = f(t_j, x_j)$ and $x_j \approx x(t_j)$. Your method should integrate at least quadratic polynomials exactly.

2. Show that the method that you derived in part a) above is consistent and stable.

3. Find conditions on α to ensure that the iteration $x_{n+1} = x_n - \alpha f(x_n)$ will converge linearly to a zero r of f if the initial point x_0 is sufficiently close to r.