Preliminary Examination in Numerical Analysis

June 1, 2012

Instructions:

1. The examination is for 3 hours.
2. The examination consists of two parts, each consisting of six equally-weighted problems:
   Part I: Matrix Theory and Numerical Linear Algebra
   Part II: Introductory Numerical Analysis
3. You may omit one problem from Part I and one problem from Part II.

PART 2 - Numerical Analysis

Problem 7. Suppose that $p \in \mathbb{P}_3$ satisfies $p(0) = f(0)$ and $p^{(j)}(1) = f^{(j)}(1)$ for $0 \le j \le 2$, where $\|f^{(n)}\|_{\infty,[0,1]} \le 2^n$ for $n \ge 0$. Provide the sharpest possible upper bounds, based on the given information, for each of the following

$$|f^{(3)}(1) - p^{(3)}(1)| \quad , \quad \left| \int_0^1 (f(x) - p(x))\, dx \right| .$$

Problem 8. Letting $f_j = f(t_j, y_j)$, consider the linear multi-step method

$$y_{n+1} - 2y_n + y_{n-1} = \frac{h}{3}(f_{n+1} + 3f_n - 3f_{n-1} - f_{n-2}) .$$

Is the method stable? Is it consistent?

Problem 9. The Trapezoid method for approximating the solution of the initial value problem,

$$y' = f(t, y) \text{ in } [a, b] \quad , \quad y(a) = \gamma ,$$

on the mesh is given by

$$y_0 = \gamma \quad , \quad y_{n+1} = y_n + \frac{h}{2}(f(t_{n+1}, y_{n+1}) + f(t_n, y_n)) .$$

Prove that this method is second-order, provided y is smooth enough. What is the minimal smoothness of y which *guarantees* this order?

Problem 10. The polynomials

$$p_0 = 1 \quad , \quad p_1 = \sqrt{2}\, x \quad , \quad p_2 = 2\sqrt{3}(x^2 - 1/2) ,$$

form an orthonormal basis for $\mathbb{P}_2$ with respect to the inner-product $\langle u, v\rangle = \int_{-1}^{1} u(x)v(x)\,|x|\, dx$. Let $\|\cdot\|$ be the induced norm. Determine the unique polynomial $p \in \mathbb{P}_2$ such that

$$\|f - p\| = \min_{q \in \mathbb{P}_2} \|f - q\| ,$$

where $f(x) = x^3$.

Problem 11. Let $g(x) = \dfrac{R^2x^3 + 3x}{3R^2x^2 + 1}$ for some $R > 0$. Show that, if $x_0 > 1/R$ and $x_{n+1} = g(x_n)$ for $n \ge 0$, then $x_n \to 1/R$, and the convergence is cubic.