Preliminary Examination in Numerical Analysis

June 3, 2013

Instructions:

1. The examination is for 3 hours.
2. The examination consists of two parts, each consisting of six equally-weighted problems:
   Part I: Matrix Theory and Numerical Linear Algebra
   Part II: Introductory Numerical Analysis
3. You may omit one problem from Part I and one problem from Part II.

PART I - Matrix Theory and Numerical Linear Algebra

Problem 1. Assume that A and $A - \delta A$ are invertible and $\|A^{-1}\|\|\delta A\| < 1/2$. Prove that

$$\frac{\|(A-\delta A)^{-1} - A^{-1}(A+\delta A)\,A^{-1}\|}{\|A^{-1}\|} \le 2\left(\kappa(A)\frac{\|\delta A\|}{\|A\|}\right)^2,$$

where $\|\cdot\|$ is a matrix operator (or subordinate matrix) norm and $\kappa(A) = \|A\|\|A^{-1}\|$.

Problem 2. Let U be a nonsingular upper triangular matrix. Let $\hat{x}$ be the computed solution to $Ux = y$ in a floating point arithmetic using backward substitution. Prove that $(U + \delta U)\hat{x} = y$ for some δU with $|\delta U| \le (n\epsilon + O(\epsilon^2))|U|$.
(You may use the fact that $fl(\sum_{i=1}^d a_i b_i) = \sum_{i=1}^d a_i b_i (1+\delta_i)$ with $\delta_i \le d\epsilon + O(\epsilon^2)$.)

Problem 3. Let A be an $m \times n$ full rank matrix with $m > n$ and let

$$A = QR = Q\begin{pmatrix} R_1 \\ 0 \end{pmatrix}$$

be its QR factorization, where R_1 is an $n \times n$ upper triangular matrix. For an $m \times k$ matrix B, derive a method to solve

$$\min_{X \in R^{n\times k}} \|AX - B\|_F$$

using the given QR factorization.

Problem 4. Let $A \in R^{m\times n}$ and $b \in R^m$ $(m \ge n)$. Let $A = U\Sigma V^T$ be the singular value decomposition of A, where $U = [u_1, u_2, \cdots, u_m]$ and $V = [v_1, v_2, \cdots, v_n]$ are orthogonal and

$$\Sigma := \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \in R^{m\times n};\ \Sigma_1 := \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{pmatrix}$$

with $\sigma_1 \ge \cdots \ge \sigma_k > 0$. Determine when $Ax = b$ has no solution, exactly one solution, or infinitely many solutions. Write down the solution or the solution set when it exists.

Problem 5. Assume that $A \in R^{n\times n}$ is diagonalizable, i.e. $A = V\Lambda V^{-1}$ where $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ with $|\lambda_1| > |\lambda_2| \ge \cdots \ge |\lambda_n|$. If $x_0 \in R^n$ is such that $V^{-1}x_0 = [\alpha_1, \alpha_2, \cdots, \alpha_n]^T$ with $\alpha_1 \ne 0$, prove that the sequence of vectors generated by the power method with x_0 as the initial vector converges in direction to the eigenvector corresponding to λ_1.

Problem 6. For $A_0 \in R^{n\times n}$ and $\sigma \in C$, consider the double shift QR iteration:

$$\begin{aligned} A_0 - \sigma I &= Q_1 R_1, \\ A_1 &= R_1 Q_1 + \sigma I, \\ A_1 - \bar{\sigma} I &= Q_2 R_2, \\ A_2 &= R_2 Q_2 + \bar{\sigma} I. \end{aligned}$$

($\bar{\sigma}$ is the complex conjugate of σ.) Show that A_2 is similar to A_0 and

$$(Q_1 Q_2)(R_2 R_1) = A_0^2 - 2Re(\sigma)A_0 + |\sigma|^2 I\ .$$

PART 2 - Numerical Analysis

Problem 7. Suppose that $f \in C^\infty(\mathbb{R})$ has a root at r, and that $f^{(j)}(r) = 0$ for $0 \leq j < k$, but $f^{(k)}(r) \neq 0$. Show that, if x_0 is sufficiently close to r, then the modified Newton iteration,

$$x_{n+1} = x_n - \frac{kf(x_n)}{f'(x_n)} ,$$

will converge (at least) quadratically to r, with

$$\frac{x_{n+1} - r}{(x_n - r)^2} \to \frac{f^{(k+1)}(r)}{k(k+1)f^{(k)}(r)} .$$

(Note that the case $k = 1$ corresponds to the standard Newton iteration.)

Problem 8. Consider the initial value problem

$$y'(t) = f(t, y(t)) \text{ for } t \in [a, b] \quad , \quad y(a) = \gamma ,$$

which is discretized using the classical fourth-order Runge-Kutta method (RK4):

$$\begin{aligned}
K_1 &= f(t_k, y_k) \\
K_2 &= f(t_k + h/2, y_k + hK_1/2) \\
K_3 &= f(t_k + h/2, y_k + hK_2/2) \\
K_4 &= f(t_k + h, y_k + hK_3) \\
y_{k+1} &= y_k + h(K_1 + 2K_2 + 2K_3 + K_4)/6
\end{aligned}$$

where $h = (b - a)/N$ and $t_k = a + kh$, $0 \leq k \leq N$.

1. Show that, if $f(t, y) = g(t)$, then RK4 can be derived using Simpson's rule to approximate the integral in

$$y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_{k+1}} g(t)\, dt .$$

2. Assuming that g is sufficiently smooth, show that, if $y_k = y(t_k)$, then $|y(t_{k+1}) - y_{k+1}| = \mathcal{O}(h^5)$.

Problem 9. Suppose $\omega \in C(a, b)$ is such that

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,\omega(x)\, dx$$

defines an inner-product on $C[a, b]$, and that $\{p_n : n \geq 0\}$ is a family of polynomials which are orthogonal with respect to $\langle \cdot, \cdot \rangle$, with $\deg(p_n) = n$. Let $\{x_k : 1 \leq k \leq n\}$ be the (distinct) roots of p_n, and define

$$w_k = \int_a^b \ell_k(x)\omega(x)\, dx \text{ where } \ell_k(x) = \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{x - x_j}{x_k - x_j} .$$

Prove that, for any polynomial f of degree $\leq 2n-1$,

$$\int_a^b f(x)\omega(x)\,dx = \sum_{k=1}^{n} w_k f(x_k)\,.$$

Problem 10. Suppose that $y \in C^2[a,b]$ satisfies

$$y'(t) = f(t, y(t)) \text{ for } t \in [a,b] \quad , \quad y(a) = \gamma\,,$$

where $|f(t,v) - f(t,w)| \leq L|v-w|$ on $R = [a,b] \times \mathbb{R}$ for some finite $L > 0$. Given $h = (b-a)/N$ and $t_k = a + kh$, $0 \leq k \leq N$, we define $y_0 = \gamma$ and $y_{k+1} = y_k + hf(t_{k+1}, y_{k+1})$ for $0 \leq k < N$. Argue that, if $Lh < 1$,

$$|y(t_k) - y_k| \leq \left(\frac{(1-Lh)^{-k} - 1}{2L} \|y''\|_{L^\infty[a,b]}\right) h \quad \text{for} \quad 0 \leq k \leq N\,.$$

Problem 11. Let $g(x) = \cos(x) + 1/2$.

1. Show that there is an r such that, if any $x_0 \in \mathbb{R}$ is given and $x_{n+1} = g(x_n)$ for $n \geq 0$, then $x_n \to r$.
2. Provide a specific $\gamma \in (0,1)$ such that $|x_{n+1} - r| \leq \gamma |x_n - r|$ for n sufficiently large, regardless of the choice of x_0.

Problem 12. Below are convergence histories, given in columns, of three different approximations of $\int_{-1}^{1} f(x)\,dx$ using a uniform composite rule of mesh-size h:

A) The midpoint rule when $f \in C^4[-1,1]$.

B) Simpson's rule when $f(x) = \sqrt{3-x^2}$.

C) Simpson's rule when $f(x) = \sqrt{1-x^2}$.

The first column contains the mesh-size h. Label each of the final three columns with the method (A,B,C) which best fits the observed convergence behavior. You do not need to show work for this problem.

h			
2.0000e+00	2.3746e-01	8.4426e-03	1.7355e-01
1.0000e+00	8.2762e-02	8.1470e-04	4.6537e-02
5.0000e-01	2.8999e-02	6.2395e-05	1.1917e-02
2.5000e-01	1.0202e-02	4.1940e-06	2.9998e-03
1.2500e-01	3.5975e-03	2.6774e-07	7.5130e-04
6.2500e-02	1.2702e-03	1.6827e-08	1.8791e-04
3.1250e-02	4.4879e-04	1.0532e-09	4.6983e-05
1.5625e-02	1.5862e-04	6.5848e-11	1.1746e-05
7.8125e-03	5.6070e-05	4.1163e-12	2.9365e-06
3.9062e-03	1.9822e-05	2.5890e-13	7.3414e-07
1.9531e-03	7.0079e-06	1.6875e-14	1.8353e-07