Preliminary Examination in Numerical Analysis

June. 9, 2014

Instructions:

1. The examination is for 3 hours.
2. The examination consists of two parts, each consisting of six equally-weighted problems:
   Part I: Matrix Theory and Numerical Linear Algebra
   Part II: Introductory Numerical Analysis
3. You may omit one problem from Part I and one problem from Part II.

PART I - Matrix Theory and Numerical Linear Algebra

Problem 1. Let $X = [x_{ij}] \in R^{n\times n}$ be such that $\|X\|_{\max} := \max_{i,j}|x_{ij}| < 1/n$. Prove that $I - X$ is invertible and $\|(I-X)^{-1}\|_{\max} \le 1/(1-n\|X\|_{\max})$.

Problem 2. (8 points) Let $A = LU$ be the LU-factorization of A with $|l_{ij}| \le 1$. Prove that $\|U\|_{\max} \le 2^{n-1}\|A\|_{\max}$, where $\|A\|_{\max} = \max_{i,j}|a_{ij}|$.

Problem 3. Let $\mathrm{fl}\,(e)$ denote the computational result of an expression e in a floating point arithmetic and let ϵ be the machine roundoff unit. Let A and Q be two $n \times n$ matrices, and assume that Q is orthogonal. Prove that

$$\mathrm{fl}\,(AQ) = (A+E)Q, \quad \|E\|_2 \le n^3\epsilon\|A\|_2 + \mathcal{O}(\epsilon^2).$$

(You may use without proof that $|fl(x^Ty) - x^Ty| \le |x|^T|y|\delta$ with $\delta \le n\varepsilon + \mathcal{O}(\varepsilon^2)$, $\|A\|_2 \le \sqrt{n}\|A\|_1$ and $\|A\|_1 \le \sqrt{n}\|A\|_2$.)

Problem 4. Let $A \in R^{m\times n}$ and $b \in R^m$ $(m \ge n)$. Let $A = U\Sigma V^T$ be the singular value decomposition of A, where

$$\Sigma := \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \in R^{m\times n}; \text{ and } \Sigma_1 := \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{pmatrix}$$

with $\sigma_1 \ge \cdots \ge \sigma_k > 0$. Determine when the least squares problem

$$\min_{x\in R^n} \|Ax - b\|_2. \tag{1}$$

has exactly one solution, or infinitely many solutions. Write down the solution or the solution set when it exists.

Problem 5. Let μ be an approximate eigenvalue of an $n \times n$ matrix A and x be an approximate eigenvector with $\|x\|_2 = 1$. Let $r = Ax - \mu x$. Show that there is an $n \times n$ matrix E with $\|E\|_2 = \|r\|_2$ such that μ is an eigenvalue of $A + E$ with x a corresponding eigenvector.

Problem 6. Consider the shifted QR algorithm for $A \in R^{n\times n}$:

Algorithm:
    $A_0 = A$
    For $i = 0, 1, 2, \cdots, m$
        Factorize $A_i - \mu_i I = Q_i R_i$ (QR-factorization)
        $A_{i+1} = R_i Q_i + \mu_i I$
    End

Prove that $\Pi_{i=0}^m (A - \mu_i I) = (Q_0 Q_1 \cdots Q_m)(R_m \cdots R_1 R_0)$.

PART 2 - Numerical Analysis

Problem 7. Find the degree two polynomial p_2 that interpolates $f(x) = \ln(x)$ at $x_0 = 1$, $x_1 = 2$ and $x_2 = 3$. For a point $t \in [1, 3]$ give two expressions for the interpolation error $f(t) - p_2(t)$. Your first expression should involve a divided difference, and your second expression should involve some unknown point $\xi \in [a, b]$. Use one of your expression to provide a bound for the interpolation error on $[x_0, x_2]$.

Problem 8. Let $f \in C^1[a, b]$ have a Lipschitz continuous first derivative, i.e., $|f'(x) - f'(y)| \leq L|x - y|$ for some $L > 0$. Suppose there exists an $\alpha \in (a, b)$ such that $f(\alpha) = 0$. Show that there exists a $\delta > 0$ such that for any starting value $x_0 \in [\alpha - \delta, \alpha + \delta]$, Newton's method will converge quadratically to α.

Problem 9. Consider a fixed point iteration $x_{n+1} = g(x_n)$, with g having a Taylor series $g(x) = \alpha + c_1(x - \alpha) + c_2(x - \alpha)^2 + \cdots$, where α is a fixed point of g. What must be true of c_1 to ensure convergence of the fixed point iteration to α for an initial x_0 sufficiently close to α? Define a procedure similar to Richardson extrapolation for generating a sequence of iterates that converge quadratically to α.

Problem 10. Consider approximating the derivative of a function $f(x)$ at $x = 0$ by approximating an integral of the form $I = \int_{-h}^{h} tf(t)dt$ with a small fixed $h < 1$. Verify that the integral has the expansion $I = c_1 h^3 f'(0) + c_2 h^5 f^{(3)}(0) + \cdots$ and write down c_1 and c_2. Then $\frac{1}{c_1 h^3} I$ would form an $O(h^2)$ approximation to $f'(0)$. Denoting by I_n the approximation to I using an n-point quadrature rule on $[-h, h]$ with equidistant points, what should n be in order for $\frac{1}{c_1 h^3} I_n$ to achieve an $O(h^2)$ approximation to $f'(0)$? (Note: The n-point quadrature rule here refers to the quadrature rule derived from the interpolating polynomial at n points. It is *not* the composite quadrature rule.)

Problem 11. Find a nonzero polynomial q of degree two that can be used to find the nodes x_i in the Gaussian quadrature formula

$$\int_0^\infty f(x)w(x)dx \approx \sum_{i=1}^{2} w_i f(x_i),$$

where $w(x) = e^{-x}$. The approximation should be exact for all polynomials up to degree 3. State what condition q must satisfy.

Problem 12. Show that the following multistep method is consistent,

$$y_{n+1} = -9y_n + 10y_{n-1} + \frac{h}{2}(13f(x_n, y_n) + 9f(x_{n-1}, y_{n-1})).$$

Is the method stable? For $f(x, y) = 0$ and and initial conditions $y_0 = 1$, $y_1 = 1$, give an expression for y_i. Is this example consistent with your conclusions regarding the stability?