Preliminary Examination in Numerical Analysis

May 29, 2015

Instructions:

1. The examination is for 3 hours.
2. The examination consists of ten equally-weighted problems. The first five cover Matrix Theory and Numerical Linear Algebra and the last five cover Introductory Numerical Analysis
3. You may **omit one** problem (i.e. work nine out of the ten problems).

Problem 1. Let $L = [l_{ij}]$ be an $n \times n$ lower triangular matrix with the diagonals equal to 1. Write down the forward substitution algorithm for solving $Lx = b$. Prove that the computed solution $\hat{x}$ satisfies $(L + \delta L)\hat{x} = b$ with $|\delta L| \leq (n-1)\epsilon|L| + \mathcal{O}(\epsilon^2)$.
(You may use without proof that $\mathrm{fl}\left(\sum_{i=1}^n x_i y_i\right) = \sum_{i=1}^n x_i y_i (1 + \delta_i)$ with $|\delta_i| \leq n\epsilon + \mathcal{O}(\epsilon^2)$.)

Problem 2. Let A be an invertible upper triangular matrix and let $X = [x_{ij}] \in R^{n \times n}$ be an upper triangular matrix with all the diagonal entries being zero. Prove that $A - X$ is invertible and

$$\|(A - X)^{-1}\| \leq \sum_{i=0}^{n-1} \|A^{-1}\|^{i+1} \|X\|^i.$$

Problem 3. Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $\delta b \in \mathbb{R}^m$. Assume that $\operatorname{rank}(A) = n$. If x is the solution to the least squares problem $\min_{x \in \mathbb{R}^n} \|Ax - b\|_2$ and $\hat{x}$ is the solution to the perturbed problem $\min_{x \in \mathbb{R}^n} \|Ax - b - \delta b\|_2$, prove that

$$\frac{\|\hat{x} - x\|_2}{\|x\|_2} \leq \kappa_2(A) \frac{\|\delta b\|_2}{\|b\|_2}$$

where $\kappa_2(A) = \|A\|_2 \|A^+\|_2$.

Problem 4. Let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$ be the singular values of $A \in \mathbb{R}^{m \times n}$. Prove that

$$\sigma_1 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \quad \text{and} \quad \sigma_n = \min_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}.$$

Problem 5. Let $H = [h_{ij}]$ be an upper Hessenberg matrix with $h_{i+1,i} \neq 0$ for $1 \leq i \leq n-1$ and let $H = QR$ with $R = [r_{ij}]$ be its QR factorization. Assume that H is singular and H_1 is obtained from H after one iteration of the QR algorithm. Prove that $r_{n,n} = 0$ and the last row of H_1 is entirely zero.

Problem 6. Consider the following interpolation problem: given $f(x)$ and $n+1$ distinct points $x_0, x_1, \cdots, x_n$, find $c_0, c_1, \cdots, c_n$ such that

$$p_n(x) = \sum_{j=0}^{n} c_j e^{jx}, \text{ satisfies } p_n(x_i) = f(x_i), i = 0, 1, \ldots n.$$

Show that the interpolation function $p_n(x)$ can always be constructed by relating it to the standard polynomial interpolation problem.

Problem 7. Given that a fixed point iteration

$$x_{n+1} = g(x_n)$$

converges to a fixed point α with order $p > 1$ for all x_0 in some neighborhood of α, i.e.

$$(x_{n+1} - \alpha) = c_0(x_n - \alpha)^p + c_1(x_n - \alpha)^{p+1} + \ldots$$

show that with three successive values of the fixed point iteration x_{n-1}, x_n and x_{n+1}, one can use extrapolation to determine a sequence that asymptotically will converge to α with order $p+1$.

Problem 8. Given a quadrature rule for approximating

$$\int_0^h f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i)$$

with nodes $x_i \in (0, h)$ and weights w_i for $i = 0, 1, \cdots, n$ such that the rule is exact for polynomials only up to degree p, where $n \leq p \leq 2n+1$. Show that one can construct a corrected quadrature rule, i.e. there exists an α such that

$$\sum_{i=0}^{n} w_i f(x_i) + \alpha \left(f^{(p)}(h) - f^{(p)}(0) \right) h^p$$

such that this new rule has precision of at least $p+1$. (*Hint*: Your expressions should work for the corrected trapezoid and midpoint rules for which $p = 2$ and the corresponding α's are $-1/12$ and $1/24$.)

Problem 9. State the conditions under which Newton's method applied to a function $f(x)$ defined on an interval (a, b), will converge to a root $\alpha \in (a, b)$ of f. Assuming that the conditions you stated are met, show that $f(x_i)$ will converge quadratically to zero with asymptotic error constant $\frac{f''(\alpha)}{2(f'(\alpha))^2}$.

Problem 10. The following mutlistep method was derived using a Hermite interpolant

$$y_{n+2} + 4y_{n+1} - 5y_n = h\left(2f(x_n, y_n) + 4f(x_{n+1}, y_{n+1})\right)$$

Show that the method is consistent but not zero-stable. In floating point arithmetic what is likely to happen when the multistep method is used to solve $y' = f(x, y)$ where $f(x, y) = 0$ and $y_0 = 1/10$, and $y_1 = 1/10$ are used as the initial conditions for the method.