Preliminary Examination in Numerical Analysis

June 3, 2016

Instructions:

1. The examination is for 3 hours.
2. The examination consists of ten equally-weighted problems. The first five cover Matrix Theory and Numerical Linear Algebra and the last five cover Introductory Numerical Analysis
3. You may **omit one** problem (i.e. work nine out of the ten problems).

Unless the problem tells you otherwise, you may assume that all norms are 2-norms.

Problem 1. Show for $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times \ell}$ that in floating point arithmetic

$$\frac{\|AB - \mathrm{fl}\,(AB)\|_F}{\|AB\|_F} \leq \kappa(A) n \|A\|_F \varepsilon + O(\varepsilon^2),$$

where $\|A\|_F = (\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2)^{1/2}$ denotes the Frobenius norm of the matrix, ε denotes machine precision, and $\kappa(A) = \|A\|_F \|A^{-1}\|_F$ denotes the Frobenius norm condition number of A. You may use without proof the following backward error result $\mathrm{fl}\,(x^T y) = (x+e)^T y$ where $|e_i| \leq n\varepsilon |x_i| + O(\varepsilon^2)$.

Problem 2. Show that for $A \in \mathbb{R}^{n\times n}$, $x, y \in \mathbb{R}^n$ and $\|A^{-1}xy^T\| < 1$

$$\|(A + xy^T)^{-1} - A^{-1}\| \leq \frac{\|A^{-1}x\|\|A^{-T}y\|}{1 - |y^T A^{-1} x|}.$$

Problem 3. Using a QR factorization give an expression for the minimum norm solution to the rank deficient least squares problem

$$\min_{x\in\mathbb{R}^n} \|Ax - b\|,$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $\mathrm{rank}(A) = r < m, n$. You may assume the r leading columns of A have full rank.

Problem 4. Show that the numerical radius $r(A)$ of a matrix A is a norm, where

$$r(A) = \max\{|x^* A x| : x \in \mathbb{C}^n, \|x\| = 1\}.$$

Problem 5. For $A \in \mathbb{R}^{m\times n}$, consider the unit vector $v \in \mathbb{R}^n$ that maximizes $\frac{\|Ax\|}{\|x\|}$. Let $u = \frac{Av}{\|Av\|}$. Show that the orthogonal matrices $V = [v\ \widehat{V}]$ and $U = [u\ \widehat{U}]$ block diagonalize A, that is

$$U^T A V = \begin{pmatrix} \sigma & \mathbf{0} \\ \mathbf{0} & * \end{pmatrix}$$

Problem 6. Let r be a positive real number and let $\{x_n\}_0^\infty$ be the sequence of iterates obtained for approximating $\sqrt{r}$ by Newton's method for $x^2 = r$.

a) Simplify Newton's recursive relation for $\{x_n\}_0^\infty$.

b) Let $e_n = x_n - \sqrt{r}$. Find and simplify a recursive relation for $\{e_n\}_0^\infty$.

c) If $x_0 > \sqrt{r}$, show that $\{x_n\}_0^\infty$ is decreasing and converges quadratically. p

d) If $0 < x_0 < \sqrt{r}$, show that $\{x_n\}_0^\infty$ converges quadratically.

Problem 7. Assume f is continuous on $[a, b]$. Let p_{n-1} be the least squares approximation to f in the norm $\|g\|_2 = \left(\int_a^b g^2(x)\omega(x)\,dx\right)^{1/2}$ from polynomials of degree $n-1$, where $\omega(x)$ is a positive weight function. Prove that there exist at least n points $x_i \in [a,b]$ such that $p_{n-1}(x_i) = f(x_i)$. (Hint: assume the contrary and consider the function $e(x) = p_{n-1}(x) - f(x)$.)

Problem 8. Given $p > -1$, find constants A and B such that

$$\int_0^1 x^p f(x)\,dx = Af(0) + Bf(1) + E(f)$$

holds with $E(f) = 0$ when f is a linear function. Find an explicit expression for the error function $E(f)$ when f has a continuous second derivative on $[0,1]$.

Problem 9. For $x_1, x_2, \cdots, x_n \in [-1,1]$, consider the quadrature rule

$$\int_{-1}^1 f(x)dx \approx w_0 f(-1) + \sum_{i=1}^n w_i f(x_i) + w_{n+1} f(1).$$

If $p_n \in \mathbb{P}_n$ is the nth orthogonal polynomial in the inner product $(f,g)_\omega$ with $\omega(x) = 1 - x^2$, i.e. $(p_n, f)_\omega = 0$ for any $f \in \mathbb{P}_{n-1}$, and if $x_1, x_2, \cdots, x_n$ are the roots of p_n, prove that the quadrature rule is exact on $\mathbb{P}_{2n+1}$. What quadrature rule do you get when $n = 1$?

Problem 10. Find A, B, C so that the linear multistep method of the form

$$x_n = x_{n-1} + h[Af_n + Bf_{n-1} + Cf_{n-2}]$$

has the highest order of approximation possible. Determine whether the resulting method is convergent.