Preliminary Examination in Numerical Analysis

June 6, 2018

Instructions:

1. The examination is for 3 hours.
2. The examination consists of eight equally-weighted problems. The first four cover Matrix Theory and Numerical Linear Algebra and the last four cover Introductory Numerical Analysis.
3. Attempt all problems.

Problem 1. Let A be a real $n \times n$ matrix.

a) State the Cauchy-Schwarz inequality for $n \times 1$ matrices x and y.

b) Define $\|A\|_2$.

c) Apply (a) and (b) to prove that $\|A^T\|_2 = \|A\|_2$ and $\|A^T A\|_2 = \|A\|_2^2$. (Hint: Start by showing that $\|Ax\|_2^2 \leq \|A^T A\|_2 \|x\|_2^2$ for all $n \times 1$ matrices x.)

Problem 2. Let $A \in \mathbb{R}^{m \times n}$ have full column rank and have the QR decomposition $A = QR$ where $Q \in \mathbb{R}^{m \times n}$ and $R \in \mathbb{R}^{n \times n}$. For any $b \in \mathbb{R}^m$, show that $x = R^{-1}Q^T b$ is the solution to the least squares problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2.$$

(You are not allowed to use the normal equation.)

Problem 3. Let A_0 be a given real matrix with real eigenvalues that have distinct absolute values. The following is an outline of the algorithm of QR iteration with a shift:

i = 0
repeat
  Choose a shift σ_i near an eigenvalue of A_i
  Use the QR decomposition to factor $A_i - \sigma_i I = Q_i R_i$
  Let $A_{i+1} = R_i Q_i + \sigma_i I$
  $i = i + 1$
until convergence of A_i's

a) Show that A_i and A_{i+1} have the same eigenvalues.

b) Describe as explicitly as possible the matrix A_i that is obtained when the algorithm converges.

c) Describe what the algorithm of "chasing the buldge" has to do with the the above algorithm.

Problem 4. Let A be an $n \times n$ matrix with $A = [a_{ij}]$, where the a_{ij}'s may be real or complex numbers.

a) Derive an explicit expression for $\text{tr}(A^*A)$ in terms of the a_{ij}'s.

b) Use the Schur decomposition to show that if $\lambda_1 \ldots \lambda_n$ are the eigenvalues of A repeated according to multipicity, then

$$\sum_{j=1}^{n} |\lambda_j|^2 \leq \|A\|_F^2.$$

c) Show that if equality holds in the above inequality, then A is normal.

Problem 5. Let $x_1, x_2, \cdots, x_n$ be machine numbers. Consider computing $\sum_{i=1}^n x_i^2$ by the algorithm

$$p_1 = x_1^2; \quad p_k = p_{k-1} + x_k^2; \text{ for } k = 2, 3, \cdots, n.$$

Find an upper bound for the relative error of the computed p_n in terms of the machine precision ϵ and n. (You can ignore the higher order terms.)

Problem 6. Let n and k be integers with $0 \leq k \leq n$ and let $\omega(t)$ be a polynomial of degree n having distinct real roots $t_1, \ldots, t_n$ satisfying

$$\int_a^b \omega(t)q(t)\,dt = 0$$

for all polynomials q of degree up to $k-1$, where $a < b$. Show that there exist real numbers $w_1, \ldots, w_n$ such that

$$\int_a^b p(t)\,dt = \sum_{j=1}^n w_j p(t_j)$$

holds for all polynomials $p(t)$ of degree up to $n+k-1$.

Problem 7. Consider the fixed point iteration $x_{n+1} = g(x_n)$ for $g(x) = 1 + \sin(x)$.

1. Show that $g(x)$ has a fixed point $r \in (\pi/2,\ 5\pi/6)$.
2. For any $x_0 \in R$, prove that there exists $\gamma \in (0,1)$ such that $|x_{n+1} - r| \leq \gamma |x_n - r|$ for all $n \geq 2$.

Problem 8. Consider the fourth-order Runge-Kutta method (RK4)

$$\begin{aligned}
K_1 &= f(t_k, y_k)\\
K_2 &= f(t_k + h/2, y_k + hK_1/2)\\
K_3 &= f(t_k + h/2, y_k + hK_2/2)\\
K_4 &= f(t_k + h, y_k + hK_3)\\
y_{k+1} &= y_k + h(K_1 + 2K_2 + 2K_3 + K_4)/6
\end{aligned}$$

for the initial value problem $y'(t) = f(t, y(t))$, $y(0) = y_0$. Prove that, if $f(t, y) = g(t)$ and $g(t) \in C^4(R)$, the local truncation error is $O(h^4)$.