Preliminary Examination in Numerical Analysis

June 11, 2021

Instructions:

1. The examination is for 3 hours.
2. The examination consists of eight equally-weighted problems.
3. Attempt all problems.

Problem 1. Let $\mathbf{x} \in \mathbb{R}^n$. Show that, in floating point arithmetic, $\mathrm{fl}(\mathbf{x}^T\mathbf{x}) = \widehat{\mathbf{x}}^T\widehat{\mathbf{x}}$ for some vector $\widehat{\mathbf{x}}$. Find an upper bound for the backward error $\|\widehat{\mathbf{x}} - \mathbf{x}\|_2$. (Ignore second order terms of machine precision ε).

Problem 2. Let $A \in \mathbb{C}^{m\times n}$ with $m \geq n$ be a full rank matrix. Consider the block 2-by-2 system

$$\begin{bmatrix} I & A \\ A^* & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix},$$

where I is an m-by-m identity matrix.

(a) Show that this system has a unique solution $(\mathbf{r}, \mathbf{x})$ and $\mathbf{x}$ solves the linear least squares problem $\min_{\mathbf{x}\in\mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|_2^2$.

(b) Find the eigenvalues of $\begin{bmatrix} I & A \\ A^* & \mathbf{0} \end{bmatrix}$ in terms of the singular values of A.

Problem 3. Let $U \in \mathbb{C}^{n\times n}$ be a nonsingular upper triangular matrix.

(a) Show that $U + U^{-1}$ is nonsingular.

(b) Design an algorithm to solve the linear system $(U + U^{-1})\mathbf{x} = \mathbf{b}$ in $\mathcal{O}(n^2)$ operations. Note that computing U^{-1} requires $\mathcal{O}(n^3)$ operations. (*Hint:* use backward substitution.)

Problem 4. Let $A = S + i\mathbf{u}\mathbf{u}^T \in \mathbb{R}^{n\times n}$ where $S \in \mathbb{R}^{n\times n}$ is a symmetric matrix, $\mathbf{u} \in \mathbb{R}^n$, and i is the unit imaginary number. Show how to compute an orthogonal matrix Q such that $Q^TAQ = T + \sigma\mathbf{e}_1\mathbf{e}_1^T$, where T is a symmetric triadiagonal matrix, σ is a scalar, and $\mathbf{e}_1$ is the first column of I_n.

Problem 5. Let f be a positive function defined on $[-1, 1]$ and assume

$$\min_{-1\leq x\leq 1} |f(x)| = m_0, \qquad \max_{-1\leq x\leq 1} |f^{(k)}(x)| = M_k, \quad k = 0, 1, \ldots.$$

Let $p_{n-1}(f; \cdot)$ be the Lagrange polynomial of degree $\leq n-1$ interpolating f at the n Chebyshev points $x_k = \cos(\frac{2k-1}{2n}\pi)$ for $k = 1, \ldots, n$.

(a) Estimate the maximum relative error $r_n = \max_{-1\le x\le 1} \left|\frac{f(x)-p_{n-1}(f;x)}{f(x)}\right|$.

(b) Apply the result to $f(x) = e^{2x}$.

Problem 6. Use Newton's interpolation formula to derive a quadrature rule of the form

$$\int_0^1 f(x)x^\alpha dx = c_0 f(0) + c_1 f(1) + c_2 f'(0) + E(f), \quad \alpha > -1.$$

Find c_0, c_1, c_2 and an expression for $E(f)$, and specify the degree of exactness.

Problem 7. Consider the following Runge-Kutta method

$$y_{i+1} = y_i + a_1 h f(t_i, y_i) + a_2 h f(t_i + \alpha h, y_i + \beta h f(t_i, y_i)), \quad i = 0, \dots, N-1.$$

Show that this method cannot have local truncation error $\mathcal{O}(h^3)$ for any choice of the constants a_1, a_2, α and β.

Problem 8. Consider the following linear multistep method

$$y_{i+2} = 2y_{i+1} - y_i + h[f(t_{i+1}, y_{i+1}) - f(t_i, y_i)], \quad i = 0, 1, \dots, N-2.$$

Analyze this method for its order, stability, convergence and region of absolute stability.