Preliminary Examination in Numerical Analysis

June 3, 2022

Instructions:

1. The examination is for 3 hours.
2. The examination consists of eight equally-weighted problems.
3. Attempt all problems.

Problem 1. Consider an $n \times n$ tridiagonal matrix

$$A = \begin{bmatrix} h & -h/4 & 0 & 0 & \cdots & 0 \\ -h/4 & h & -h/4 & 0 & \cdots & 0 \\ & & & \cdots & \cdots & \\ 0 & \cdots & 0 & -h/4 & h & -h/4 \\ 0 & \cdots & 0 & 0 & -h/4 & h \end{bmatrix}$$

where $h > 0$. Show that $\kappa_\infty(A) := \|A\|_\infty \|A^{-1}\|_\infty \leq 3$.

Problem 2. Consider the least squares problem of the form

$$\min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|_2, \quad \text{where } A = UV.$$

Here $U \in \mathbb{R}^{m \times r}$ has orthonormal columns and $V \in \mathbb{R}^{r \times n}$ has rank r with $r < n \leq m$. Show that the solution of the problem is not unique, and then use the QR factorization of V to find the general solution.

Problem 3. Assume that $A \in \mathbb{R}^{m \times n}$ with $\text{rank}(A) = n < m$. Use the SVD of A to solve $\min\limits_{\|\mathbf{x}\|_2 = 1} \mathbf{x}^T H \mathbf{x}$ where $H = \begin{bmatrix} I_n & A^T \\ A & I_m \end{bmatrix}$.

Problem 4. If λ is an eigenvalue of the matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ with $a_{21} \neq 0$, prove that one step of the single shifted QR iteration with λ as the shift produces an upper triangular matrix with the (2,2) entry being λ.

Problem 5. Assume that $f \in C^2[0, 3]$ and M is the maximum value of $|f''(x)|$ on $[0, 3]$. Show that

$$\left| \int_0^3 f(x)dx - \frac{3}{2}\left(f(1) + f(2)\right) \right| \leq \frac{11}{12} M$$

Problem 6. Consider the following iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n)}{2f'(x_n)} \left(\frac{f(x_n)}{f'(x_n)} \right)^2$$

for solving the equation $f(x) = 0$. Assuming α is a simple root of the equation, and x_n converges to α as $n \to \infty$. Prove that the method converges cubically.

Problem 7. Consider approximating $f(x)$ on $[-1, 1]$ by a cubic interpolating polynomial $p(x)$ at the four Chebyshev nodes on $[-1, 1]$. Prove that $\max_{-1 \le x \le 1} |f(x) - p(x)| \le \frac{1}{192} \|f^{(4)}\|_\infty$. (Hint: $\cos 3\pi/8 = \sin \pi/8$.)

Problem 8.

a) Consider the multistep method of the form

$$y_{n+3} + y_{n+2} + \alpha \left(y_{n+1} + y_n \right) = h \left(\beta f_{n+2} + \gamma f_{n+1} \right)$$

Find parameters α, β, γ so that the method has order $p = 2$.

b) Discuss the stability properties of the method obtain in (a).