Preliminary Examination in Numerical Analysis

June 2nd, 2023

Instructions:

1. The examination is for 3 hours.
2. The examination consists of eight equally-weighted problems.
3. Attempt all problems.

Problem 1. Show that if an $n \times n$ matrix E satisfies $\|E\| < 1$, then

$$\|(I+E)(I-E)^{-1} - I - 2E\| \leq \frac{2\|E\|^2}{1-\|E\|}$$

where $\|\cdot\|$ is any matrix operator norm.

Problem 2. Let $A \in \mathbb{R}^{m\times n}$ with $m \geq n > r = \text{rank}(A)$, $\mathbf{b} \in \mathbb{R}^m$, and $C \in \mathbb{R}^{n\times n}$ be symmetric positive definite. Consider the minimization problem $\min_{\mathbf{x}\in\mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|_2^2 + \lambda \mathbf{x}^T C\mathbf{x}$.
(a) If $\lambda > 0$, then find the normal equation.
(b) If $\lambda = 0$, then use the singular value decomposition $A = U\Sigma V^T$ to find all the solutions of the problem.

Problem 3. Let $A = U\Sigma V^T \in R^{m\times n}$ be the singular value decomposition of A, where $m \geq n$ and Σ is $n \times n$. If $\|A^T A - I_n\|_2 = \epsilon < 1$, prove that $\|A - UV^T\|_2 \leq \epsilon$.

Problem 4. Let μ_1 and μ_2 be two approximate eigenvalues of an $n \times n$ matrix A and let x_1 and x_2 be two corresponding approximate eigenvectors. Assume x_1 and x_2 are orthogonal to each other and $\|x_1\|_2 = \|x_2\|_2 = 1$. Let $R = AX - XD$ where $X = [x_1, x_2]$ and $D = \begin{pmatrix} \mu_1 & \\ & \mu_2 \end{pmatrix}$. Prove that there exists an $n \times n$ matrix E with $\|E\|_2 \leq \|R\|_2$ such that μ_1 and μ_2 are the eigenvalues of $A+E$ with x_1 and x_2 as corresponding eigenvectors.

Problem 5. Construct the weighted Newton-Cotes formula

$$\int_0^1 f(x)x^2 dx = a_0 f(0) + a_1 f(\frac{1}{2}) + a_2 f(1) + E(f),$$

and derive an expression for $E(f)$ in terms of f. Specify the degree of exactness, i.e., the largest integer d such that $E(x^d) = 0$.

Problem 6. Suppose that $f \in C^n([a,b])$. Let p_{n-1}, a polynomial of degree at most $n-1$, be the Lagrange interpolation polynomial of f, with interpolation points

$$x_k = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)\cos\left(\frac{2k-1}{2n}\pi\right), \quad k = 1, \dots, n.$$

Show that

$$|f(x) - p_{n-1}(x)| \leq \frac{1}{2^{n-1}n!}\left(\frac{b-a}{2}\right)^n \max_{\xi\in[a,b]}\left|f^{(n)}(\xi)\right|$$

(Hint: Use Chebyshev nodes)

Problem 7. When f is a smooth function, prove the following forward-difference formula

$$f'(x) = \frac{f(x-h) - 8f\left(x-\frac{h}{2}\right) + 8f\left(x+\frac{h}{2}\right) - f(x+h)}{6h} + O\left(h^4\right)$$

Problem 8. Consider the one-step method for the initial value problem $y' = f(t,y), y(0) = y_0$,

$$y_{n+1} = y_n + h\left[(1-\theta)f(t_n, y_n) + \theta f(t_{n+1}, y_{n+1})\right], \quad n = 0, 1, \ldots,$$

where $\theta \in [0,1]$ is a parameter. Find all θ so that the method is A-stable.