Preliminary Examination in Numerical Analysis

May 31, 2024

Instructions:

1. The examination is for 3 hours.
2. The examination consists of eight equally-weighted problems.
3. Attempt all problems.

Problem 1. Let $U \in \mathbb{R}^{n\times n}$ be a nonsingular upper triangular matrix and solve $U\mathbf{x} = \mathbf{b}$ by backward substitution. Show that barring overflow or underflow, the computed solution $\widehat{\mathbf{x}}$ satisfies $(U + \delta U)\widehat{\mathbf{x}} = \mathbf{b}$ with $|\delta U| \leq (n-1)\varepsilon|U| + O(\varepsilon^2)$ and ε is the machine precision. Hint: You may use without proof that $fl\,(\sum_{i=1}^{n} x_i y_i) = \sum_{i=1}^{n} x_i y_i(1+\delta_i)$ with $|\delta_i| \leq n\varepsilon + \mathcal{O}(\varepsilon^2)$.

Problem 2. For $\lambda > 0$, consider the following regularized least squares problem

$$\min_{x\in\mathbb{R}^n} \| Ax - b \|_2^2 + \lambda \| x \|_2^2 .$$

The solution satisfy the normal equation: $x = (A^TA + \lambda I)^{-1}A^Tb$. Let A have SVD $A = U\Sigma V^T$, where $\Sigma \in \mathbb{R}^{n\times n}$, $\Sigma = diag(\sigma_1, \cdots, \sigma_n)$, $U = [u_1, \cdots, u_n]$, and $V = [v_1, \cdots, v_n]$. Show that $Ax = \sum_{j=1}^{n} u_j \frac{\sigma_j^2}{\sigma_j^2 + \lambda} u_j^T b$.

Problem 3. Let $A \in \mathbb{R}^{n\times n}$ and $H = \begin{bmatrix} I_n & -A^T \\ -A & I_n \end{bmatrix}$ be nonsingular where I_n is the n-by-n identity matrix.

(a) Find the eigenvalues of H in terms of the singular values of A.

(b) Compute the condition number $\kappa_2(H)$ in terms of singular values of A.

Problem 4. Let A_0 be a given real matrix with real eigenvalues that have distinct absolute values. The following is an outline of the algorithm of QR iteration with a shift:

$i = 0$
repeat
    Choose a shift σ_i near an eigenvalue of A_i
    Use the QR decomposition to factor $A_i - \sigma_i I = Q_i R_i$
    Let $A_{i+1} = R_i Q_i + \sigma_i I$
    $i = i + 1$
until convergence of A_i's

(a) Show that A_i and A_{i+1} have the same eigenvalues.

(b) Describe as explicitly as possible the matrix A_i that is obtained when the algorithm converges.

Problem 5. Construct the natural cubic spline function with knots at -1, 0, and 1 that interpolates the given data points:

t	-1	0	1
y	1	2	5

Problem 6. Define the *order of convergence* for a sequence $\{x_n\}_{n=0}^{\infty}$ that converges to α. Next, find out both the order of convergence and the associated asymptotic error constant for Newton's method when used to find the root α of a smooth function $f(x)$ satisfying $f(\alpha) = f'(\alpha) = 0$.

Problem 7. Construct the following weighted Newton-Cotes formula

$$\int_0^1 f(x) \cdot \frac{1}{\sqrt{x}}\, dx = a_0 \cdot f(0) + a_1 \cdot f(1) + E(f),$$

and find an expression of the error function $E(f)$ in terms of an appropriate derivative of f.

Problem 8. Consider the following one-step implicit method for the initial value problem $x'(t) = f(t, x)$ with $x(a) = x_0$ and $t \in [a, b]$:

$$x_{n+1} = x_n + \frac{1}{2} h \left(f(t_n, x_n) + f(t_{n+1}, x_{n+1}) \right).$$

(a) Determine the order of local truncation error for this method.

(b) Explain A-stability and whether this method is A-stable.