Preliminary Examination on Partial Differential Equations

January 3, 2002

Instructions

This is a three-hour examination. You are to work a total of **five problems**. The exam is divided into two parts. **You must do at least two problems from each part.**

Please indicate clearly on your test papers which five problems are to be graded.

You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

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PART ONE

Problem 1. Let $u \in C^2(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is open. Suppose $\Delta u \geq 0$ in Ω, i.e., u is subharmonic. Show that

$$u(x) \leq \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\,dy \quad \text{for all } B(x,r) \subset \Omega.$$

Problem 2. Let B_r denote the ball in $\mathbb{R}^n$ centered at 0 with radius r. Let u be a nonnegative harmonic function in B_4. Prove the following version of Harnack inequality: for any $x, y \in B_1$,

$$3^{-n}u(x) \leq u(y) \leq 3^n u(x).$$

Problem 3. Let Ω be a bounded smooth domain in $\mathbb{R}^n$. Suppose that $u \in C^2(\overline{\Omega} \times [0,\infty))$ is a solution to the initial-Neumann problem

$$\begin{cases} u_t - \Delta u = 0 \quad \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = g(x), \quad x \in \Omega \end{cases}$$

where ν denotes the outward unit normal to $\partial\Omega$, and g satisfies

$$\frac{\partial g}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

Show that, for any $0 < T < \infty$,

$$\int_\Omega |u(x,T)|^2 dx + \int_0^T \int_\Omega |\nabla u(x,t)|^2 dx dt = \int_\Omega |g(x)|^2 dx.$$

Problem 4. Suppose $u \in C^2(\mathbb{R}^n \times [0,\infty))$ solves

$$u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty).$$

Fix $x_0 \in \mathbb{R}^n$, $t_0 > 0$ and consider the cone

$$C = \{(x,t) : \ 0 \leq t \leq t_0, \ |x - x_0| \leq t_0 - t\}.$$

Show that, if $u(x,0) = u_t(x,0) = 0$ on $B(x_0,t_0)$, then $u \equiv 0$ within the cone C.

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PART TWO

Problem 5. Let Ω be a bounded domain in $\mathbb{R}^n$.

(a). Let $u, v \in L^1_{loc}(\Omega)$ and α be a multi-index. What does it mean if v is said to be the αth-weak partial derivative of u.

(b). State the definitions of $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ for $1 \le p < \infty$.

(c). Use definitions in (a) and (b) to show that, if $\psi \in C_c^\infty(\Omega)$ and $u \in W^{1,2}(\Omega)$, then

$$D^\alpha(\psi u) = \psi D^\alpha u + u D^\alpha \psi \quad \text{for } |\alpha| = 1$$

and $\psi u \in W_0^{1,2}(\Omega)$.

Problem 6. Let Ω be a bounded domain in $\mathbb{R}^n$. Consider the second-order, uniform elliptic partial differential operator L given by

$$L = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i}\right) + \sum_{k=1}^{n} b_k \frac{\partial}{\partial x_k} + c,$$

where $a_{ij} = a_{ji}$, b_k, $c \in L^\infty(\Omega)$. Let $B[\cdot,\cdot]$ be the associated bilinear form on $H_0^1(\Omega)$.

(a). Prove that there exist positive constants C_1, C_2, C_3 such that, for any $u, v \in H_0^1(\Omega)$,

$$|B[u,v]| \le C_1 \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)},$$

and

$$C_2 \|u\|_{H_0^1(\Omega)} \le B[u,u] + C_3 \|u\|_{L^2(\Omega)}.$$

(b). If $b_k = 0$ for $1 \le k \le n$ and $c = 0$, prove that the following Dirichlet problem for L has exactly one weak solution for any $f \in L^2(\Omega)$:

$$Lu = f \ \text{ in } \Omega, \quad \text{and } u = 0 \ \text{ on } \partial\Omega.$$

Problem 7. Let Ω be a bounded domain in $\mathbb{R}^n$. Let

$$\mathcal{L} = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{k=1}^{n} b_k \frac{\partial}{\partial x_k}$$

where $a_{ij} = a_{ji}$, b_k are continuous and satisfy the uniform ellipticity condition on Ω. Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $\mathcal{L}u < 0$ in Ω. Show that

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

This is a weak version of the maximum principle.

Problem 8. Let

$$L = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)$$

be a uniform elliptic operator on Ω with $a_{ij} \in L^\infty(\Omega)$, $a_{ij} = a_{ji}$. Suppose $u \in H^1(\Omega)$ is a weak solution of $Lu = 0$ in Ω. Show that

$$\int_{B(x_0,r)} |\nabla u(x)|^2\, dx \le \frac{C}{r^2}\int_{B(x_0,2r)} |u(x)|^2\, dx$$

for any $B(x_0, 2r) \subset\subset \Omega$, where C depends only on n, $\|a_{ij}\|_\infty$, and the ellipticity constants of L. This is the Caccippoli inequality.