Partial Differential Equations Preliminary Examination January 2003

Instructions

This is a three-hour examination. You need to solve a total of **five problems**. The exam is divided into two parts. **You must do at least two problems from each part**.

Please indicate clearly on your test papers that which five problems are to be graded.

You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

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PART ONE

(1) Find a solution $u(x,y)$ of the problem

$$u_x u_y = u \quad \text{in } \mathbb{R}\times\mathbb{R}$$
$$u(0,y) = y^2 \quad \text{for each } y \in \mathbb{R}$$

(2) Let u be a solution to the heat equation $u_t = u_{xx}$ in the unit square $S = \{(x,t) : 0 < x,t < 1\}$ with boundary values given by $u(0,t) = u(1,t) = 0$ for $0 \le t \le 1$ and $u(x,0) = 1$, $u(x,1) = 0$ for $0 \le x \le 1$. Show that

$$\int_S u_x^2(x,t)\,dxdt = \frac{1}{2}.$$

(3) Let $\Omega_T = \Omega \times (0,T]$ where $T > 0$ and Ω is a bounded open set in $\mathbb{R}^n$ with smooth boundary. Show that there exists at most one function $u \in C^2(\overline{\Omega_T})$ which solves

$$\begin{cases} u_{tt} - \Delta u = f & \text{in } \Omega_T, \\ u = g & \text{on } (\partial\Omega \times [0,T]) \cup (\Omega \times \{t=0\}), \\ u_t = h \text{ on } \Omega \times \{t=0\}. \end{cases}$$

(4) Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose $F : \mathbb{R} \to \mathbb{R}$ is C^1 with F' bounded. Show that, if $u \in W^{1,p}(\Omega)$ for some $1 < p < \infty$, then $F(u) \in W^{1,p}(\Omega)$.

PART TWO

(5) Let $Lu = \Delta u + \sum_{i=1}^n b_i(x) D_i u + c(x)u$. Prove that if $c < 0$ and is bounded in $\Omega \subset \mathbb{R}^n$ and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies $Lu = f$ in Ω then

$$\sup_{\Omega} |u| \le \sup_{\partial\Omega} |u| + \sup_{\Omega} \left|\frac{f}{c}\right|.$$

(6) Suppose $u \in H^1(\Omega)$ where Ω is a bounded open set in $\mathbb{R}^n$. Show that

$$\int_\Omega Du \cdot D\varphi \, dx = 0$$

for each $\varphi \in C_c^\infty(\Omega)$ if, and only if,

$$\int_\Omega |Du|^2 \, dx \le \int_\Omega |Dv|^2 \, dx$$

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for each $v \in H^1(\Omega)$ such that $u - v \in H_0^1(\Omega)$.

(7) Suppose Ω is a bounded open subset of $\mathbb{R}^n$ and $u \in C^0(\Omega)$ satisfies

$$\int_\Omega u \Delta \varphi \, dx = 0$$

for each $\varphi \in C_c^\infty(\Omega)$. Show that, in fact, $u \in C^2(\Omega)$ and

$$\Delta u = 0$$

in Ω.

(8) Let

$$L = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)$$

be a uniformly elliptic operator on Ω with $a_{ij} \in L^\infty(\Omega)$, $a_{ij} = a_{ji}$. Suppose $u \in H^1(\Omega)$ is a weak solution of $Lu = 0$ in Ω. Show that

$$\int_{B(x_0,r)} |Du(x)|^2 \, dx \le \frac{C}{(R-r)^2} \inf_{t \in \mathbb{R}} \int_{B(x_0,R)} |u(x) - t|^2 \, dx$$

for any $B(x_0, R) \subset \Omega$, where $0 < r < R$ and C depends only on n, $\|a_{ij}\|_\infty$, and the ellipticity constant of L.