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Preliminary Examination
Partial Differential Equations
January 2004

Instructions

This is a three-hour examination. You need to solve a total of **five problems**. The exam is divided into two parts. **You must do at least two problems from each part.**

Please indicate clearly on your test papers which five problems are to be graded.

You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

PART ONE

(1) Let $f(x) = |x|$ for $|x| \leq 1$ and $f(x) = 1$ for $|x| \geq 1$.

(a) Express in terms of a " Poisson ' integral the solution u to the Dirichlet problem for harmonic functions in $H = \{(x, y) : y > 0\} \subset \mathbf{R}^2$ with $u = f$ continously on the boundary of H.

(b) Use your answer to (a) to show that $\lim_{y \to 0} u(0, y)/y \to \infty$ (so u_y does not extend boundedly to the closure of H).

(2) Given $u(x, t) = \dfrac{t}{4\pi} \displaystyle\int_{|\omega|=1} h(x + t\,\omega)\, dS$ where $x = (x_1, x_2, x_3) \in \mathbf{R}^3, t \in (0, \infty)$, $dS =$ surface area on the unit sphere in $\mathbf{R}^3$ and h is twice continuously differentiable on $\mathbf{R}^3$. Show that

(a) If $t > 0$, then $u(y, t) \to 0$ and $u_t(y, t) \to h(x)$ as $(y, t) \to (x, 0)$.

(b) u is a solution to the wave equation in $U = \mathbf{R}^3 \times (0, \infty)$ (i.e $u_{tt} = \Delta u$ in U where Δ is the Laplacian of u in the space variable x).

Hint: In (b) use the identity

$$\int_{B(x,t)} \Delta u \, dx = \int_{\partial B(x,t)} u_\nu \, dS$$

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where $B(x,t) = \{y : |y-x| < t\}$, dx is Lebesgue three measure, dS = surface area, and ν is the outer unit normal to $\partial B(x,t)$.

(3) Use the method of characteristics to solve the quasilinear PDE:

$$u_x + 2y\, u_y = u^2, \quad u(0,y) = y.$$

(4) Let $\Phi(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$ when $t > 0$ and $\Phi(x,t) = 0$ otherwise when $(x,t) \in \mathbf{R^n} \times \mathbf{R} \setminus \{(0,\ldots,0,0)\}$. Given that Φ is a solution to the heat equation in $\mathbf{R^n} \times \mathbf{R} \setminus \{(0,\ldots,0,0)\}$ and that $\int_{\mathbf{R^n}} \Phi(x,t)dx = 1$ (dx = Lebesgue n measure) whenever $t > 0$. Show that Φ is a fundamental solution to the heat equation with pole at $(0,\ldots,0,0)$ in the sense that

$$\int_{\mathbf{R^n}\times\mathbf{R}} (\Delta\psi + \psi_t)\, \Phi(x,t)\, dxdt = \psi(0,\ldots,0,0)$$

whenever ψ is infinitely differentiable with compact support.

PART TWO

(5) For a domain $\Omega \subset R^n$,
  (a) state the definition of the Sobolev space $W^{1,p}(\Omega)$ for $1 \le p < \infty$, and
  (b) use the integration by parts to prove the following interpolation inequality:

$$\int_{R^n} |\nabla u|^p \le C(n)(\int_{R^n} |u|^p)^{\frac{1}{2}}(\int_{R^n} |\nabla^2 u|^p)^{\frac{1}{2}}$$

  where $u \in W^{2,p}(R^n)$ and $C(n) > 0$ depends only on n.

(6) For $1 \le i,j \le n$, let $a_{ij} \in C^0(\Omega)$ and define

$$L = -\sum_{ij=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}), \quad x \in \Omega$$

  (a) State the uniform ellipticity condition for L.
  (b) Suppose that $f \in H^{-1}(\Omega)$. State the definition of $u \in H_0^1(\Omega)$ being a weak solution of the boundary value problem:

$$\begin{aligned} Lu &= f, \quad \text{in } \Omega \\ u &= 0, \quad \text{on } \partial\Omega \end{aligned}$$

  and prove that if L is uniformly elliptic, then there exists at most one solution $u \in H_0^1(\Omega)$ of the problem.

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(c) Use the difference quotient method to prove: if $(a_{ij}) \in C^1(\Omega)$ and $f \in L^2(\Omega)$, then any weak solution $u \in H^1(\Omega)$ of the equation $Lu = f$ is in $H^2_{\text{loc}}(\Omega)$. Moreover, for any open subset $U \subset\subset \Omega$,

$$\|D^2u\|_{L^2(U)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)})$$

where C does not depend on u or f.

(7) Suppose $u \in H^1(\Omega)$ where Ω is a bounded open set in $\mathbb{R}^n$. Show that

$$\int_\Omega Du \cdot D\varphi \, dx = 0$$

for each $\varphi \in C_0^\infty(\Omega)$ if, and only if,

$$\int_\Omega |Du|^2 \, dx \leq \int_\Omega |Dv|^2 \, dx$$

for each $v \in H^1(\Omega)$ such that $u - v \in H_0^1(\Omega)$.

(8) Let L be the uniformly elliptic operator in problem 6. Prove the Caccioppoli inequality: if $u \in H^1(\Omega)$ is a weak solution to $Lu = 0$ on Ω. Then, for any ball $B_R \subset \Omega$, one has

$$\int_{B_r} |Du|^2 \leq \frac{C}{(R-r)^2} \inf_{c \in R} \int_{B_R} |u - c|^2$$

for any $0 < r < R$.