Preliminary Examination
Partial Differential Equations
January 2005

Instructions

This is a three-hour examination. You need to solve a total of **five problems**. The exam is divided into two parts. **You must do at least two problems from each part.**

Please indicate clearly on your test papers which five problems are to be graded. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

PART ONE

(1) Let $\Gamma(x,t) = (4\pi t)^{-n/2}\exp(-|x|^2/4t)$, $t > 0$ and $x \in \mathbf{R^n}$. If f is continuous and bounded on $\mathbf{R^n}$, define u by

$$u(x,t) = \int_{\mathbf{R}^n} \Gamma(x-y,t) f(y)\, dy.$$

It is obvious that u is infinitely differentiable on $\mathbf{R^n} \times (0,\infty)$. Show u extends continuously to the closure of this set. That is, show

$$\lim_{(y,s)\to(x,0)} u(y,s) = f(x) \text{ whenever } x \in \mathbf{R^n}.$$

You may use the fact that

$$\int_{\mathbf{R}^n} \exp(-|x|^2/4)\, dx = (4\pi)^{n/2}.$$

(2) Let $B = \{x \in \mathbf{R^n} : |x| < 1\}$. Suppose

$$u \in C^2(B) \cap C^1(\overline{B})$$
$$u = 0 \text{ on } \partial B$$
$$|\Delta u| \le M \text{ on } B$$

where M is a positive constant.

Show that

$$\left|\frac{\partial u}{\partial \nu}\right| \le \frac{M}{n} \text{ on } \partial B$$

where, for $x \in \partial B$, $\frac{\partial u}{\partial \nu}$ denotes the directional derivative of u at x in the direction of the inward normal to ∂B.

Hint: Note that if $w(x) = 1 - |x|^2$, then $\Delta w = -2n$ on B and $w = 0$ on ∂B.

(3) For a given smooth function $f \in C_0^\infty(\mathbf{R} \times [0, \infty))$ with compact support, suppose that $u \in C_0^2(\mathbf{R} \times [0, \infty))$ is a C^2-solution, with compact support, of the following one dimensional non-homogenous wave equation:

$$u_{tt} - u_{xx} = f, \quad (x,t) \in \mathbf{R} \times [0, \infty).$$

Set $E(t) = \int_{\mathbf{R}} (|u_t|^2 + |u_x|^2)(x,t)\, dx$ for $t \geq 0$. Show that

$$E(t) \leq e^t\, E(0) + e^t \int_0^t (e^{-s} \int_{\mathbf{R}} |f|^2(x,s)\, dx)\, ds$$

holds for all $t \geq 0$.

(4) Solve the following 1st order PDE by the method of characteristics:

$$\begin{aligned} u_x + u_y &= u^2,\ y > 0, \\ u(x,0) &= g(x),\ x \in \mathbf{R}, \end{aligned}$$

where $g : \mathbf{R} \to \mathbf{R}$ is a given positive C^2 function.

PART TWO

(5) Given an open set $\Omega \subset \mathbf{R^n}$.
  (a) State the definition of the Sobolev space $W^{1,p}(\Omega)$ whenever $1 \leq p < \infty$.
  (b) Prove that if Ω is bounded, then $W^{1,q}(\Omega) \subset W^{1,p}(\Omega)$ whenever $1 \leq q \leq p < \infty$.
  (c) For fixed $p, 1 < p < \infty$, show (by way of an example) that there exists $u \in W^{1,q}(B)$ for $q < p$ but $u \notin W^{1,p}(B)$. Here B denotes the open ball in $\mathbf{R^n}$ with radius 1 and center at 0.

(6) Given that $C^\infty(\Omega)$ is dense in $W^{1,p}(\Omega)$ whenever $\Omega \subset \mathbf{R^n}$ is an open set and $1 \leq p < \infty$. Let $u \in W^{1,p}(\Omega)$ for fixed p, $1 \leq p < \infty$.
  (a) If $F \in C^1(\mathbf{R})$ with $F(0) = 0$ and $|F'| \leq M < \infty$ show that $F \circ u \in W^{1,p}(\Omega)$.
  (b) Use (a) to show that $u^+ = \max(u, 0) \in W^{1,p}(\Omega)$.

(7) For $1 \le i, j \le n$, let $(a_{ij}(x))$ be positive definite almost everywhere in the open set $\Omega \subset \mathbf{R}^n$ and $a_{ij} \in L^\infty(\Omega)$. Put

$$L = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}), \quad x \in \Omega.$$

(a) Given $u, v \in H^1(\Omega)$, what is meant by u is a weak solution and v a weak subsolution to L in Ω?
(b) Show that if $a_{ij} \in C^1(\Omega)$ and $u \in C^2(\Omega)$ is a weak solution to $Lu = 0$ in Ω, then $Lu = 0$ pointwise in Ω.
(c) Show that if u, L are as in (b) and $\phi \in C^\infty(\mathbf{R})$ is convex (i.e $\phi'' \ge 0$), then $v = \phi \circ u$ is a weak subsolution to L in Ω.

(8) Let $\Omega \subset \mathbf{R}^n$ be a bounded open set, $n > 2$, and $f \in L^2(\mathbf{R}^n)$. Put

$$u(x) = \int_\Omega f(y)\,|x-y|^{2-n}\,dy \text{ whenever } x \in \Omega.$$

(a) Show that $u \in H^1(\Omega)$.
(b) Prove for some constant c that

$$\int_\Omega \nabla u \cdot \nabla \psi\,dx = c \int f\,\psi\,dx \text{ whenever } \psi \in C_0^\infty(\Omega),$$

where $\nabla \psi$ denotes the gradient of ψ.