Preliminary Examination in Partial Differential Equations

3 January 2007

Instructions

This is a three-hour examination. The exam is divided into two parts. You should attempt at least two questions from each part and a total of five questions. Please indicate clearly on your test paper which five questions are to be graded.

Provide complete solutions to each problem and give as much detail as possible. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly the theorems and definitions you are using.

PART I

1. Let u be harmonic in $B(0,1) = \{x : |x| < 1\} \subset \mathbf{R}^n$. Thus, all the second partial derivatives of u are continuous in $B(0,1)$ and $\Delta u = 0$ in $B(0,1)$.

   (a) State and prove the mean value property for u in $B(0,1)$.

   (b) Use this mean value property to show that if u has an absolute maximum in $B(0,1)$ then $u \equiv$ constant.

2. Given that derivatives of harmonic functions are harmonic.

   (a) Show that if v is harmonic in $B(0,1) = \{x : |x| < 1\}$, with $|v| \leq 1$ in $B(0,1)$, then $|\nabla v(0)| \leq c$ where c is an absolute constant and ∇v denotes the gradient of v.

   (b) Use your result in (a) and the translation and dilation invariance of harmonic functions to show that the only bounded harmonic functions in $\mathbf{R}^n$ are constants.

3. Given $u(x,t) = x\,t^{-3/2} \exp\left[-x^2/4t\right]$ when $x > 0$, $t > 0$.

   (a) Prove that $u_t = u_{xx}$ when $x > 0$, $t > 0$.

   (b) Find $\lim_{x \to 0+} u(x,t)$ when $t > 0$ and $\lim_{t \to 0+} u(x,t)$ when $x > 0$.

   (c) Does the boundary value problem

$$\begin{cases} v_t = v_{xx} & \text{when } x > 0, t > 0 \\ v(x,0) = 0 & \text{when } x > 0, t = 0 \\ v(0,t) = 0 & \text{when } x = 0, t > 0 \end{cases}$$

   have a unique solution? Why or why not?

4. Let $f \in C_0^\infty(\mathbf{R}^2)$ where $C_0^\infty(\mathbf{R}^2)$ is the collection of functions which are smooth and compactly supported in $\mathbf{R}^2$.

   Define u by

$$u(x,t) = \int_0^\infty \int_0^\infty f(x-y, t-s)\, dy\, ds.$$

   Show that u is a solution of the equation

$$u_{xt} = f.$$

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PART II

1. Let $B = B(0,1) = \{x \in \mathbf{R}^3 : |x| \le 1\}$ be the unit ball in $\mathbf{R}^3$.

   (a) For $1 \le p < \infty$, give the definition of the Sobolev space $W_0^{1,p}(B)$.

   (b) If f is in $W_0^{1,2}(B)$ and g is in $W^{1,2}(B)$, show that fg lies in $W_0^{1,3/2}(B)$.

2. Let $\Omega \subset \mathbf{R}^n$ be a bounded open set. Let A be a symmetric matrix-valued function $A(x) = (a_{ij}(x))$ with each $a_{ij} \in L^\infty(\Omega)$ and real valued. Assume that the matrix A is uniformly positive definite in Ω. Thus, there is a $\lambda > 0$ so that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge \lambda|\xi|^2, \qquad x \in \Omega,\ \xi \in \mathbf{R}^n.$$

   Define an operator by

$$L = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}),\ x \in \Omega.$$

   (a) Let $f \in H^{-1}(\Omega)$, the dual of $H_0^1(\Omega)$ (or $W_0^{1,2}(\Omega)$). Give the definition of a weak solution to

$$Lu = f, \qquad \text{in } \Omega.$$

   (b) Use the difference quotient method to prove: If each $a_{ij} \in C^1(\Omega)$ and $f \in L^2(\Omega)$, then any weak solution $u \in H^1(\Omega)$ to $Lu = f$ in Ω is in $H^2_{loc}(\Omega)$. Moreover, for any ball $B \subset\subset \Omega$,

$$\|\nabla^2 u\|_{L^2(B)} \le C(\|f\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)})$$

   where C depends only on $n, L, \operatorname{dist}(B, \partial\Omega)$. Here, $\nabla^2 u$ is the matrix of second derivatives.

3. Let L be as in problem 2. Suppose that u is a weak solution of the equation

$$Lu - u = 0$$

   in $B(0,2)$. (Note that we do not assume that u vanishes on the boundary of $B(0,2)$.)

   Show that u satisfies the Caccioppoli inequality

$$\int_{B(0,1/2)} |\nabla u|^2\, dx \le C \int_{B(0,1)} u^2\, dx.$$

   The constant C depends only on the coefficients L and the dimension, n.

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4. Assume that L and Ω are as in Problem 2 and that the coefficients of L are smooth.

Let $u \in C^\infty(\bar{\Omega} \times [0, +\infty))$ be a solution of

$$\begin{cases} u_t + Lu = 0 & \text{in } \Omega \times (0, +\infty) \\ u = 0 & \text{on } \partial\Omega \times [0, +\infty) \\ u = g & \text{on } \Omega \times \{t = 0\}. \end{cases}$$

Prove the exponential decay estimate:

$$\|u(\cdot, t)\|_{L^2(\Omega)} \le e^{-Ct} \|g\|_{L^2(\Omega)}, \qquad t > 0,$$

where $C > 0$ depends only n, Ω, L. (Hint: Multiply by u, integrate on Ω, apply the ellipticity of L and the Poincaré inequality.)

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