1/08

Preliminary Examination in Partial Differential Equations

7 January 2008, 9 a.m.–12 noon

Instructions

This is a three-hour examination. The exam is divided into two parts. You should attempt at least two questions from each part and a total of five questions. Please indicate clearly on your test paper which five questions are to be graded.

Provide complete solutions to each problem and give as much detail as possible. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly the theorems and definitions you are using.

Part I

1. Find all functions $F : \mathbf{R}^2 \to \mathbf{R}$ which have two continuous derivatives and are solutions to the equation
$$\frac{\partial^2 F}{\partial x \partial y} - 2\frac{\partial^2 F}{\partial x^2} = 0.$$
Hint: Make the change of variables $\xi = x$ and $\eta = x + 2y$.

2. Let $\Delta = \sum_{j=1}^{n} \partial^2/\partial x_j^2$ denote the Laplace operator. Suppose that $u : \mathbf{R}^3 \to \mathbf{R}$ has two continuous derivatives and $\Delta u \geq 0$. Show that
$$u(0) \leq \frac{1}{4\pi} \int_{|x|=1} u(y)\, d\sigma(y).$$

3. Let $u(x_1, x_2, \ldots, x_n) = (x_1^2 + \cdots + x_n^2)^\alpha$. For what values of α is u subharmonic in $\mathbf{R}^n \setminus \{0\}$? For which values is u superharmonic in $\mathbf{R}^n \setminus \{0\}$?

4. Let $\Omega = \{(x, y) \in R^2 : |x| \leq 1 \text{ and } |y| \leq 1\}$. Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$, $\Delta u = -1$ in Ω and $u = 0$ on $\partial\Omega$. Show that $\frac{1}{4} \leq u(0,0) \leq \frac{1}{2}$. Hint: Use the function $(x^2 + y^2)/4$.

PART II

1. For $h \neq 0$ and $i = 1, \ldots, n$, let $\Delta_i^h u$ denote the difference quotient

$$\Delta_h^i u(x) = \frac{1}{h}(u(x + he_i) - u(x)).$$

If we have $\|\Delta_i^h u\|_{L^2(\mathbf{R}^n)} \leq 1$ for $h \neq 0$ and $i = 1, \ldots, n$, show that the weak gradient, ∇u, exists and lies in $L^2(\mathbf{R}^n)$.

2. Let A be a $n \times n$ matrix-valued function on $\mathbf{R}^n$ and suppose that the entries of A are real-valued bounded measurable functions and that there is constant $\lambda > 0$ so that we have the ellipticity condition,

$$A(x)\xi \cdot \xi \geq \lambda |\xi|^2, \quad \xi \in \mathbf{R}^n \text{ and a.e. } x \in \mathbf{R}^n.$$

Let u be a function in the Sobolev space, $W^{1,2}_{loc}(\mathbf{R}^n)$ which is a weak solution of the equation $\operatorname{div} A \nabla u = 0$ in $\mathbf{R}^n$.

Show that there is a constant C so that for all $r > 0$, we have

$$\int_{\{x:|x|<r\}} |\nabla u|^2 \, dx \leq \frac{C}{r^2} \int_{\{x:|x|<2r\}} u^2 \, dx.$$

If, in addition, we assume that u is in $L^2(\mathbf{R}^n)$, conclude that u is constant.

3. Let $1 \leq p < \infty$ and suppose that u belongs to the Sobolev space $W^{1,p}(\mathbf{R}^n)$. We write points $x \in \mathbf{R}^n$ as (x', x_n) with $x' \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}$. Show that there is a constant $C = C(p)$ which depends only on p so that

$$\int_{\mathbf{R}^{n-1}} |u(x', 0)|^p \, dx' \leq C \int_{\mathbf{R}^n} |u(x)|^p + |\nabla u(x)|^p \, dx.$$

4. Let Ω be a bounded open set in $\mathbf{R}^2$ with smooth boundary. Suppose that f is in $L^2(\Omega)$. Consider the boundary value problem

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(a) Give a weak formulation of this boundary value problem.

(b) Show that you may use the Lax-Milgram theorem to establish the existence of weak solutions to this boundary value problem.

David Adams, Russell Brown, Zhongwei Shen–December 19, 2007