Preliminary Examination in Partial Differential Equations
January 7, 2009

**Instructions** This is a three-hour examination.The exam is divided into two parts. You need to solve a total of five problems. You must do at least two problems from each part. Please indicate clearly on your test papers which five problems are to be graded.

You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

## PART ONE

(1) Let $n \geq 3$. Suppose that $u$ is harmonic in $\{x \in \mathbf{R}^n : 0 < |x| < 1\}$ and continuous in $\{x \in \mathbf{R}^n : 0 < |x| \leq 1\}$. Set

$$M = \max\{u(x) : |x| = 1\}.$$

If $\limsup_{|x| \to 0} |x|^{n-2}|u(x)| = 0$, show that $u(x) \leq M$ for $0 < |x| < 1$.

(2) Let $\Omega$ be a bounded domain with smooth boundary in $\mathbf{R}^n$. Let $\Omega_T = \Omega \times (0, T)$ and let $\bar{\Omega}_T$ be the closure of $\Omega_T$. Show that there is at most one real valued function $u$ in $C^2(\bar{\Omega}_T)$ which satisfies

$$\begin{cases} \partial_t u - \Delta u = -u^3 & \text{in } \Omega \\ u(x,0) = 0, & x \in \Omega \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0. \end{cases}$$

(3) (a) Using the method of characteristics find the solution, $u(x_1, x_2)$ to $(x_1 - 1)u_{x_1} + (x_2 + 1)\, u_{x_2} = 5\, u$, with $u(x_1, 1) = 8(x_1 - 1)^2$.

(b) Explain why $u_{x_1} + u_{x_2} = x_2$, $u(x_1, x_1) = 3x_1^2$ has no solution by the method of characteristics.

(4) (a) Show that if $P, Q$ are polynomials in $x \in \mathbf{R}^n$, then the local solution $u$ to the wave equation ($u_{tt} = \Delta u$), guaranteed by Cauchy-Kowalevski about $(x, t) = 0$ satisfying $u(x, 0) = P(x)$, $u_t(x, 0) = Q(x)$, is a polynomial in $x, t$.

(b) Find a solution to the above problem when $Q(x) = |x|^2 + x_1^3$ and $P \equiv 0$.

## PART II

(5) Let $S$ be the strip $\{x \in \mathbf{R}^n : 0 < x_1 < 1\}$. Suppose that $u$ lies in the Sobolev space $W_0^{1,2}(S)$. Show that there is a constant $C$ which is independent of $u$ so that $\|u\|_{L^2(S)} \leq C\|Du\|_{L^2(S)}$.

(6) Let $B = \{y : |y| < 1\} \subset \mathbf{R}^n$.

(a) Find all $\alpha > 0$ for which $|x|^{-\alpha} \in W^{1,2}(B)$.

(b) Find all $\alpha \in (0,1)$ for which $x_1^\alpha \in W_0^{1,2}(B \cap \{y : y_1 > 0\})$.

(7) Let $\Omega \subset \mathbf{R}^n$ be a bounded domain and let $u \in W_0^{1,2}(\Omega)$ be a weak solution to

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial u}{\partial x_j}\right) = f$$

where $f \in L^2(\Omega)$, and $(A_{ij}(x))$ are bounded, measurable, with

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} A_{ij}(x)\xi_i\xi_j$$

for almost every $x \in \Omega$, some $\lambda > 0$, and all $\xi \in \mathbf{R}^n$.

(a) Define what is meant by a weak solution to the above PDE.

(b) Using (a) prove for some positive $c < \infty$ that

$$\|Du\|_{L^2(\Omega)} \leq c\|f\|_{L^2(\Omega)}.$$

(c) Using (b) find the largest $p$ guaranteed by Sobolev's theorem for which $\|u\|_{L^p(\Omega)} \leq c\|f\|_{L^2(\Omega)}$.

(8) Let $B = \{y : |y| < 1\} \subset \mathbf{R}^n$ and let $(a_{ij}(x), 1 \leq i,j \leq n)$ be uniformly elliptic and continuous on $\bar{B} = \{y : |y| \leq 1\}$. Let $v \in C^2(B) \cap C(\bar{B})$ be a solution to the nondivergence form PDE:

$$\hat{L}v = \sum_{i,j=1}^{n} a_{i,j}(x)v_{x_ix_j} = 0 \text{ for } x \text{ in } B.$$

(a) Show that if $v \leq M$ on $\partial B$, then $v \leq M$ in $B$.

(b) Show that if $\phi \in C^2(\mathbf{R})$ is convex, then $w = \phi \circ v$ is a subsolution to $\hat{L}$. That is, show $\hat{L}w \geq 0$ in $B$.