Preliminary Examination in Partial Differential Equations

8 January 2014

Instructions

This is a three-hour examination. The exam is divided into two parts. You should attempt at least two problems from each part and a total of five problems. Please indicate clearly on your test paper which five problems are to be graded.

Provide complete solutions to each problem and give as much detail as possible. More weight will be given to a complete solution to one problem than to solutions of the easy bits from two different problems. Indicate clearly the theorems and definitions you are using.

PART I

1. Assume $u \in C^2(\Omega)$ is a harmonic function. Show the gradient estimate:

$$|Du(x_0)| \leq \frac{2^{n+1}n}{\alpha(n)} \frac{1}{r^{n+1}} \int_{B_r(x_0)} |u(y)|\, dy$$

holds for any ball $B_r(x_0) \subset \Omega$. Here $\alpha(n)$ is the volume of unit ball in $\mathbf{R}^n$.

2. Given $u(x,t) = xt^{-\frac{3}{2}} \exp(-\frac{x^2}{4t})$ when $x > 0, t > 0$.

   (a) Prove that $u_t = u_{xx}$ when $x > 0, t > 0$.

   (b) Find $\lim_{x \to 0^+} u(x,t)$ when $t > 0$ and $\lim_{t \to 0^+} u(x,t)$ when $x > 0$.

   (c) Does the boundary value problem

$$\begin{aligned} v_t &= v_{xx} \text{ when } x > 0, t > 0 \\ v(x,0) &= 0 \text{ when } x > 0, t = 0 \\ v(0,t) &= 0 \text{ when} x = 0, t > 0 \end{aligned}$$

   has a unique solution? Why or why not?

3. Suppose that Ω is a bounded open subset of $\mathbf{R}^3$ with smooth boundary and let u be a $C^2(\overline{\Omega} \times [0,T)$ solution of the initial-boundary value problem

$$\begin{cases} u_{tt} - \Delta u = u_t & \text{in } \Omega \times [0,T) \\ u = 0 & \text{on } \partial\Omega \times [0,T) \end{cases}$$

   Define the energy of a solution u by

$$E(t) = \frac{1}{2} \int_\Omega u_t(x,t)^2 + |\nabla u(x,t)|^2 \, dx.$$

   Show that

$$E(t) \leq \exp(2t) E(0), \qquad t \in [0,T).$$

4. Find a solution of the initial-value problem

$$\begin{cases} u_t + uu_x = 0, & t > 0, \ x \in \mathbf{R} \\ u(x,0) = x^2, & x \in \mathbf{R} \end{cases}$$

   Explain why there is only one smooth solution of this initial-value problem.

PART II

1. Let u be in $C_c^\infty(\mathbf{R}^n)$, $n \geq 2$. Establish the representation formula,

$$u(x) = c_n \int_{\mathbf{R}^n} \frac{\nabla u(y) \cdot (x-y)}{|x-y|^n}\, dy.$$

Your proof should include a computation of the value of c_n.

2. Let $\Omega \subset \{(x_1, \ldots, x_n) \in \mathbf{R}^n : 0 < x_1 < d\}$ and $u \in C_c^\infty(\Omega)$. Show that there exists $C = C(n) > 0$ such that the following inequality

$$\|u\|_{L^2(\Omega)} \leq Cd \|\nabla u\|_{L^2(\Omega)}$$

holds (hint: it suffices to prove this when $n = 1$).

3. Suppose that $\Omega \subset \mathbf{R}^n$ is a bounded open set and $A = (a_{ij}) : \Omega \to \mathbf{R}^{n \times n}$ is a bounded measurable function that is uniformly elliptic, i.e. there exists $\lambda > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \ \forall \xi \in \mathbf{R}^n, \text{ a.e. } x \in \Omega.$$

   (a) For a given $f \in L^2(\Omega)$, define what it means for $u \in H_0^1(\Omega)$ to be a weak solution of the boundary value problem,

$$\begin{cases} -\text{div}(A\nabla u) = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

   (b) If u is a weak solution of the boundary value problem in part a) and $B(x,r) \subset \mathbf{R}^n$ is a ball, show that u satisfies the Caccioppoli inequality

$$\int_{B(x,r)\cap\Omega} |\nabla u|^2\, dy \leq C \int_{B(x,2r)\cap\Omega} \left(\frac{1}{r^2}|u|^2 + r^2|f|^2\right) dy.$$

   The constant C may depend on the coefficients and the dimension, but not the solution u.

4. Suppose that u is in $W^{1,p}(\mathbf{R})$ for some p in $(1, \infty)$. Prove that u is Hölder continuous of exponent $1 - 1/p$. More precisely, show that

$$|u(x) - u(y)| \leq |x-y|^{1-1/p}\|u'\|_{L^p(\mathbf{R})}, x, y \in \mathbf{R}.$$

Here we use u' to denote the weak derivative of u on the real line.