Preliminary Examination in Partial Differential Equations

January 2017

Instructions

This is a three-hour examination. You are to work a total of **five problems**. The exam is divided into two parts. **You must do at least two problems from each part**.

Please indicate clearly on your test papers which five problems are to be graded. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy parts from two different problems. Indicate clearly what theorems and definitions you are using.

PART ONE

Problem 1. Suppose that $u \in C^2(\mathbb{R}^n)$ is subharmonic, i.e. $\Delta u \geq 0$, where $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ is the Laplacian. Show that for all $x_0 \in \mathbb{R}^n$ and $R > 0$,

$$u(x_0) \leq \fint_{\partial B(x_0,R)} u(y)dS(y),$$

where $B(x_0, R)$ is the ball of radius $R > 0$ centered at x_0, $\partial B(x_0, R)$ is its boundary with surface measure dS , and $\fint$ is the average of the integral over the surface.

Problem 2. Let $\Phi(x)$ be the fundamental solution for the Laplacian in $\mathbb{R}^2$:

$$\Phi(x) = -\frac{1}{2\pi} \ln |x|.$$

Let $f \in C_c^\infty(\mathbb{R}^2)$ and

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)dy.$$

Show, without using distributions, that

$$-\Delta u = f.$$

Problem 3. Let Ω be a smooth bounded domain in $\mathbb{R}^n$. Show that the equation

$$\begin{aligned} u_t - \Delta u + u &= f \text{ in } \Omega \times (0, \infty) \\ u(0, x) &= g \text{ in } \Omega \end{aligned}$$

has at most one solution $u \in C^2(\overline{\Omega} \times [0, \infty))$.

Problem 4. Let $U = \{(x, y) \mid y > 0, x \in \mathbb{R}\} \subset \mathbb{R}^2$. Solve the PDE

$$\begin{cases} u_y = u^2 & (x, y) \in U \\ u(x, 0) = 1 + x^2 \end{cases} \tag{1}$$

and state clearly on what subdomain of U the solution is defined.

PART TWO

Problem 5. Fix $p \in (1, \infty)$, and suppose that $u \in W^{1,p}(\mathbb{R}^n)$ is continuous. Let $\tilde{u}$ be the restriction of u to the plane $\{x_n = 0\}$. Show that

$$\|\tilde{u}\|_{L^p(\mathbb{R}^{n-1})} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)},$$

for some constant C.

Problem 6. Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^n$, let $f \in L^2(\Omega)$, and let $V \geq 0$ be in $L^\infty(\Omega)$. Consider the boundary value problem

$$\begin{aligned} (-\Delta + V)u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

(1) State what it means for $u \in H_0^1(\Omega)$ to be a weak solution of this boundary value problem.
(2) Show that there exists a weak solution $u \in H_0^1(\Omega)$ to this boundary value problem.

Problem 7. Suppose $u \in H^1(B(0,1))$ is a weak solution to the elliptic equation

$$\sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}(x)u_{x_i}) = 0$$

on $B(0,1)$, with a_{ij} smooth on $\overline{B(0,1)}$. Let A denote the matrix (a_{ij}).

(1) Show there exists a constant C depending only on A, such that

$$\int_{B(0,1)} |\eta|^2|\nabla u|^2 dx \leq C \int_{B(0,1)} |\nabla\eta|^2|u|^2 dx$$

for each $\eta \in C_0^1(\Omega)$.

(2) Show that for each $0 < r < 1$, there exists C depending only on r and A such that

$$\int_{B(0,r)} |\nabla u|^2 dx \leq \frac{C}{(R-r)^2} \int_{B(0,1)} |u|^2.$$

Problem 8. Let Ω be a bounded smooth connected domain in $\mathbb{R}^n$.

(1) State the Rellich-Kondrachov Theorem for Ω.

(2) Fix $\alpha > 0$. Show that there exists a constant $C > 0$ depending only on α, such that for all $v \in W^{1,2}(B(0,1))$ with

$$|\{x \in B(0,1)|v(x) = 0\}| \geq \alpha,$$

we have

$$\int_\Omega v^2\,dx \leq C \int_\Omega |Dv|^2\,dx.$$

Hint: If this inequality is false, then for each $k \in \mathbb{N}$, there exists v_k with

$$\int_\Omega v_k^2\,dx \geq k \int_\Omega |Dv_k|^2\,dx.$$