Preliminary Examination in Partial Differential Equations

January 2021

Instructions

This is a three-hour examination. You are to work a total of **five problems**. The exam is divided into two parts. **You must do at least two problems from each part**.

Please indicate clearly on your test papers which five problems are to be graded. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

PART ONE

For the rest of the exam, the domain U in $\mathbb{R}^d$ is assumed to be bounded, connected, and smooth.

Problem 1.

i) Assume that $u \in C^2(\bar{U})$ is a nonnegative function that satisfies

$$\begin{cases} -\Delta u \leq 0 & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

Show that $u \equiv 0$.

ii) Show that there is at most one function $v \in C^2(\bar{U})$ that satisfies

$$\begin{cases} -\Delta v = 0 & \text{in } U \\ v = 1 & \text{on } \partial U \end{cases}$$

Problem 2. Let $U = \{(x_1, x_2), x_2 > 0\} \subset \mathbb{R}^2$. Use the method of characteristics to solve the PDE

$$\begin{cases} u_{x_1} + u_{x_2} = u & (x_1, x_2) \in U \\ u(x_1, 0) = x_1^2 & x_1 \in \mathbb{R} \end{cases}$$

Problem 3.

Let f be continuous and bounded on $\mathbb{R}^d$ and recall that

$$u(x,t) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \exp(-|y|^2/4t) f(x-y)\, dy.$$

solves the homogeneous heat equation.

Show that $\lim_{t \to 0^+} u(x,t) = f(x)$.

You may use that $\displaystyle\int_{\mathbb{R}^d} (4\pi t)^{-d/2} \exp(-|y|^2/4t)\, dy = 1$.

Problem 4.

Show that there is at most one u be in $C^\infty(\bar{U} \times [0, \infty))$ that solves the equation

$$\begin{cases} u_{tt} - \Delta u + u_t = 0, & \text{in } U \times (0, \infty) \\ u(x,0) = f(x), & x \in U \\ u_t(x,0) = 0, & x \in U \\ u(x,t) = 0, & (x,t) \in \partial U \times [0, \infty) \end{cases}$$

Hint: Consider the energy $E(t) = \int_U |Dw(x,t)|^2 + |w_t(x,t)|^2\, dx$ for a suitable function w.

PART TWO

In the following we assume that the $d \times d$ matrix $\big(a^{ij}(x)\big)$ satisfies the uniform ellipticity condition,

$$\sum_{i,j=1}^{d} a^{ij}(x)\xi_i\xi_j \geq \mu|\xi|^2 \quad \text{for a.e. } x \in \mathbb{R}^d \text{ and for any } \xi \in \mathbb{R}^d, \tag{1}$$

where $\mu > 0$. We also assume that $a^{ij}(x)$ are bounded, measurable functions.

Problem 5.

Let $r > 0$, and suppose

$$\Omega = \{(x_1, x_2, \dots, x_d) : 0 < x_1 < r\} \subset \mathbb{R}^d$$

Let $u \in C_c^\infty(\Omega)$. Show that

$$\int_\Omega u(x)^2\,dx \leq Cr^2 \int_\Omega |Du(x)|^2\,dx.$$

Here C is an absolute constant that does not depend on u or r.

Problem 6.

Consider the equation

$$\begin{cases} -\sum_{i,j=1}^{d} \left(a^{ij}(x)u_{x_i}\right)_{x_j} + b(x)\frac{\partial u}{\partial x_1} = F \text{ in } U \\ u = 0 \text{ on } \partial U \end{cases} \tag{2}$$

where $b \in L^\infty(U)$ and $F \in H^{-1}(U)$.

(a) What does it mean for $u \in H_0^1(U)$ to be a weak solution of (2)?

(b) Show that there is some $\epsilon > 0$ so that, assuming additionally that $\|b\|_{L^\infty} < \epsilon$, (2) admits a unique solution for all $F \in H^{-1}(U)$.

Problem 7.

(a) Suppose that $u \in C_c^\infty(\mathbb{R}^d)$. Using integration by parts, prove that

$$\int_{\mathbb{R}^d} |Du|^2\,dx \leq C\left(\int_{\mathbb{R}^d} u^2\,dx\right)^{1/2}\left(\int_{\mathbb{R}^d} |D^2u|^2\,dx\right)^{1/2}.$$

(b) Show that this inequality extends to $u \in H^2(\mathbb{R}^d)$.

Problem 8.

Consider the equation

$$-\sum_{i,j=1}^{d} \left(a^{ij} u_{x_i}\right)_{x_j} = 0 \tag{3}$$

.

(a) What does it mean for $u \in H^1(U)$ to be a weak solution of (3)?

(b) Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ is convex and smooth, and that $w \in H^1(U)$ is **bounded**. If $v \in C_c^\infty(U)$, show that $\tilde{v} = \phi'(w)v$ is in $H_0^1(U)$.

(c) Assume that u is a **bounded** weak solution of (3), and let ϕ as above. Show that for all $v \in C_c^\infty(U)$ so that $v \geq 0$, we have

$$-\sum_{i,j=1}^{n} \int_U a^{ij} \left(\phi(u)\right)_{x_i} v_{x_j}\, dx \leq 0$$

You may use that the chain rule also works for weak derivatives without proof.

Hint: For an arbitrary v as above, consider the test function $\tilde{v} = \phi'(u)v$.