# Preliminary Examination in Partial Differential Equations

**January 2023**

**Instructions**

This is a three-hour examination. You are to work a total of **five problems**. The exam is divided into two parts. **You must do at least two problems from each part**.

Please indicate clearly on your test papers which five problems are to be graded. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy parts from two different problems. Indicate clearly what theorems and definitions you are using.

NOTATION: For $r > 0$ and $x \in \mathbb{R}^d$, we let $B(x, r) := \{y \in \mathbb{R}^d \mid |x-y| < r\}$, the open ball of radius $r$ centered at $x \in \mathbb{R}^d$. The symbol $D_j$ denotes the partial derivative $D_j := \dfrac{\partial}{\partial x_j}$, and the symbol $D$ denotes the gradient. For a function $u(x, t)$, we write $u_t := \dfrac{\partial u}{\partial t}$ and $u_{x_j} := \dfrac{\partial u}{\partial x_j}$. The Laplacian in Cartesian coordinates is $\Delta := \sum_{j=1}^{n} \dfrac{\partial^2}{\partial x_j^2}$. The $k$-times differentiable functions of compact support on an open set $U \subset \mathbb{R}^d$ are denoted by $C_c^k(U)$.

## PART ONE

**Problem 1.** Let $u \in C^\infty(\mathbb{R}^3)$ be a harmonic function. Show that there is a constant $C > 0$ so that for all $x \in \mathbb{R}^3$ and $r > 0$,

$$|Du(x)| \leq \frac{C}{r^4} \|u\|_{L^1(B(x,r))}$$

The constant $C > 0$ is independent of $x \in \mathbb{R}^3$ and $r > 0$.

**Problem 2.** Let $f \in C(B(0,1))$, where $B(0,1) \subset \mathbb{R}^d$, is the unit ball centered at the origin. Suppose $|f(x)| \leq 1$, $\forall x \in B(0,1)$. Let $u \in C^2(B(0,1)) \cap C(\overline{B(0,1)})$ be a solution to

$$\begin{cases} -\Delta u(x) = f(x) & \text{in } B(0,1) \\ \quad u(x) = 0 & \text{on } \partial B(0,1) \end{cases}$$

Prove that for all $x \in B(0,1)$, we have

$$-\frac{1}{2d}(1 - |x|^2) \leq u(x) \leq \frac{1}{2d}(1 - |x|^2).$$

HINT: Use the maximum principle for subharmonic functions (a function $u \in C^2(B(0,1))$ is subharmonic if with $\Delta u \geq 0$), and the fact that $-\Delta(1 - |x|^2) = 2d$.

**Problem 3.** Let $f \in C_c^2(\mathbb{R})$.

a) Deduce the explicit formula for the solution to the 1-dimensional wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0, & \text{in } \mathbb{R} \times (0, \infty) \\ u = f, u_t = 0, & \text{on } \mathbb{R}^d \times \{t = 0\} \end{cases}$$

b) Show that, if $f \not\equiv 0$, then

$$\liminf_{t \to \infty} \sup_{x \in \mathbb{R}} |u(x,t)| > 0.$$

In other words, solutions to the 1-dimensional wave equation do not decay in time!

**Problem 4.** Assume that $u \in C^2(\mathbb{R}^d \times [0, \infty); \mathbb{R})$ solves the equation

$$\begin{cases} u_t - \Delta u + u^3 = 0, & \text{in } \mathbb{R}^d \times (0, \infty) \\ u = 0, & \text{on } \mathbb{R}^d \times \{t = 0\} \end{cases}$$

Assume also that, for each fixed $t \geq 0$, the function $g(x) := u(x,t)$ is compactly supported. Show that $u \equiv 0$. HINT: Use the energy method.

## PART TWO

**Problem 5.** Consider the region $\Omega = \{(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : x_d > 0\}$, the upper half-space. Prove that for $u \in C_c^\infty(\mathbb{R}^d)$, we have the inequality

$$\int_{\partial\Omega} u^2 \, d\sigma \leq C \int_{\Omega} \left[ |Du|^2 + u^2 \right] \, dx.$$

where $C$ depends on the dimension $d$, and $\sigma$ is the surface measure on $\partial\Omega = \mathbb{R}^{d-1}$.

**Problem 6.**

a) Suppose $f$ and $g$ are locally integrable functions on $\mathbb{R}^d$. Give a careful definition of what it means for $g$ to be the weak derivative of $f$ with respect to $x_i$, $D_i f = g$.

b) Suppose $f$ and $g$ are locally integrable on $\mathbb{R}^d$ and $D_i f = g$ as in part a) and that $\phi$ lies in $C^\infty(\mathbb{R}^d)$. Prove the product rule

$$D_i(\phi f) = \phi g + f D_i \phi.$$

*For the next two problems, the domain $U \subset \mathbb{R}^d$ is assumed to be an open, bounded, connected set with smooth boundary. We also assume that the $d \times d$ real, matrix-valued function $\left(a^{ij}(x)\right)$ is symmetric, with bounded measurable entries satisfying $\|a^{ij}\|_{L^\infty(U)} \leq 1/\mu$, and satisfying the uniform ellipticity condition,*

$$\mu|\xi|^2 \leq \sum_{i,j=1}^{d} a^{ij}(x)\xi_i\xi_j \quad \text{for a.e. } x \in \mathbb{R}^d \text{ and for any } \xi \in \mathbb{R}^d,$$

*where $\mu > 0$. We let $L$ denote the operator*

$$L = -\sum_{i,j=1}^{d} D_i a^{ij} D_j.$$

**Problem 7.** A real-valued function $u \in H^1(U)$ is a *weak solution* of $Lu = 0$ on $U$, without boundary conditions, if, for all $v \in H_0^1(U)$, we have $B[v, u] = 0$, where $B$ is the quadratic form associated with $L$. Prove that such a weak solution $u$ satisfies

$$\int_{B(x,r)} |(Du)(y)|^2 \, dy \leq \frac{C}{r^2} \int_{B(x,2r)} |u(y)|^2 \, dy.$$

**Problem 8.**

a) State the Lax-Milgram Theorem.

b) Let $f$ be in $L^2(U)$ and let $b \in L^\infty(U)$. Consider the boundary-value problem

$$\begin{cases} Lu + bu = f, & \text{in } U \\ u = 0, & \text{on } \partial U \end{cases}$$

Show that there is a constant $\beta > 0$ so that if $\|b\|_{L^\infty(U)} \leq \beta$, then the above boundary-value problem has a weak solution.