Preliminary Examination on Partial Differential Equations

May 30, 2001
9-12, CB 234

Instructions

This is a three-hour examination. You are to work a total of **five problems**. The exam is divided into two parts. **You must do at least two problems from each part.**

Please indicate clearly on your test papers which five problems are to be graded.

You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

Partial Differential Equations May 2001

PART ONE

Problem 1. Let Ω be an open set in $\mathbb{R}^n$. Show that, if u is harmonic in Ω, then

$$|\nabla u(x_0)| \leq \frac{C}{r^{n+1}} \int_{B(x_0,r)} |u(y)|\, dy$$

for any ball $B(x_0, r) \subset\subset \Omega$, where C depends only on n.

Problem 2. Let

$$\Gamma(x) = \frac{1}{(2-n)\omega_{n-1}|x|^{n-2}}$$

in $\mathbb{R}^n$, $n \geq 3$ where ω_{n-1} denotes the surface area of the unit sphere in $\mathbb{R}^n$. Show that, if Ω is a bounded open set with smooth boundary and $u \in C^2(\overline{\Omega})$, then

$$u(x) = \int_\Omega \Gamma(y-x)\Delta u(y)\, dy - \int_{\partial\Omega} \left\{ \Gamma(y-x)\frac{\partial u}{\partial n(y)} - u(y)\frac{\partial \Gamma(y-x)}{\partial n(y)} \right\} d\sigma(y)$$

where $n(y)$ denotes the outward unit normal to $\partial\Omega$ at y.

Problem 3. Find all solutions $u(x,t)$ of the one-dimensional heat equation $u_t = u_{xx}$ of the form

$$u(x,t) = v\left(\frac{x}{\sqrt{t}}\right), \qquad t > 0.$$

Hint: v has to satisfy a linear ordinary second-order equation.

Problem 4. Let $\Omega_T = \Omega \times (0, T]$ where $T > 0$ and Ω is a bounded open set in $\mathbb{R}^n$ with smooth boundary. Show that there exists at most one function $u \in C^2(\overline{\Omega_T})$ which solves

$$\begin{cases} u_{tt} - \Delta u = f & \text{in } \Omega_T, \\ u = g & \text{on } (\partial\Omega \times [0,T]) \cup (\Omega \times \{t=0\}), \\ u_t = h \text{ on } \Omega \times \{t = 0\}. \end{cases}$$

Partial Pleminary Exam May 2006

PART TWO

Problem 5. Show that, if $p > n$, then

$$\|u\|_{L^\infty(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for any $u \in W^{1,p}(\mathbb{R}^n)$, where C depends only on n and p. Hint: first show that

$$\frac{1}{|B(x,1)|}\int_{B(x,1)} |u(y)-u(x)|\,dy \le C\int_{B(x,1)} \frac{|\nabla u(y)|}{|y-x|^{n-1}}\,dy$$

for $u \in C^1(\mathbb{R}^n)$ by using

$$u(y)-u(x) = \int_0^1 \frac{d}{dt}u(x+t(y-x))\,dt.$$

Problem 6. Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^n$. Let

$$\mathcal{L} = -\sum_{j,k}\frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial}{\partial x_k}\right)$$

where $a_{jk} \in L^\infty(\Omega)$ and $a_{jk} = a_{kj}$.

(a). What does it mean if $\mathcal{L}$ is said to be uniformly elliptic?

(b). For a given $f \in L^2(\Omega)$. State the definition for a function $u \in H_0^1(\Omega)$ to be a weak solution of the boundary value problem:

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

(c). Prove that there exists a unique weak solution to the boundary value problem in part (b).

Problem 7. Let Ω be a bounded open set in $\mathbb{R}^n$. Suppose $u \in C^2(\Omega)\cap C(\overline{\Omega})$ satisfies

$$\Delta u + \sum_{i=1}^n a_i(x)\frac{\partial u}{\partial x_i} + c(x)\,u = 0 \quad \text{in } \Omega.$$

Assume $c(x) < 0$ in Ω and $u = 0$ on $\partial\Omega$. Show that $u = 0$.

Problem 8. Let

$$\mathcal{L} = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$$

be a uniform elliptic operator on Ω with $a_{ij} \in L^\infty(\Omega)$, $a_{ij} = a_{ji}$. Suppose $u \in H^1(\Omega)$ is a weak solution of $\mathcal{L}u = 0$ in Ω. Show that

$$\int_{B(x_0,r)} |\nabla u(x)|^2\,dx \le \frac{C}{r^2}\int_{B(x_0,2r)} |u(x)|^2\,dx$$

for any $B(x_0,2r) \subset\subset \Omega$, where C depends only on n, $\|a_{ij}\|_\infty$, and the ellipticity constants of $\mathcal{L}$. This is the Caccippoli inequality.