Preliminary Examination on Partial Differential Equations

May 29, 2002

Instructions

This is a three-hour examination. You need to solve a total of **five problems**. The exam is divided into two parts. **You must do at least two problems from each part.**

Please indicate clearly on your test papers that which five problems are to be graded.

You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

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Throughout this exam, Ω denotes a bounded open domain in $\mathbb{R}^n$ with smooth boundary.

PART ONE

Problem 1. Let $u \in C^2(\Omega)$ be a harmonic function on Ω. State and prove the mean value property for u.

Problem 2. Show that, if $u \in C^2(\Omega)$ is harmonic function on Ω, then

$$R \max_{B(x,R)} |Du| \leq C_n \inf_{c\in\mathbb{R}} \frac{1}{|B(x,2R)|} \int_{B(x,2R)} |u(y) - c|\, dy$$

for any $B(x, 2R) \subset \Omega$, where $B(x,r)$ denotes the ball centered at x with radius r.

Problem 3. Suppose $0 < T < \infty$ and $u(x,t)$ is a smooth solution of

$$\begin{aligned} u_{tt} - \Delta u = f \quad & \text{in } \Omega_T = \Omega \times (0,T) \\ u = 0 \quad & \text{on } \partial\Omega \times [0,T) \\ u = u_t = 0 \quad & \text{on } \Omega \times \{0\}. \end{aligned}$$

Show that

$$\sup_{0<t<T} \int_\Omega \left\{ |u_t(x,t)|^2 + |D_x u(x,t)|^2 \right\} dx \leq C_T \, \|f\|^2_{L^2(\Omega_T)}.$$

Problem 4. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a bounded continuous function and let $u(x,t)$ be defined by

$$u(x,t) = (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy$$

for $x \in \mathbb{R}^n$ and $t > 0$. Show that $u(x,t)$ is a solution of the problem

$$\begin{aligned} u_t - \Delta u = 0 \quad & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) \quad & \text{on } \mathbb{R}^n \end{aligned}$$

where the initial condition is satisfied in the sense that

$$\lim_{\substack{(x,t)\to(x_0,0) \\ t>0}} u(x,t) = g(x_0) \qquad \text{for each } x_0 \in \mathbb{R}^n.$$

Is $u(x,t)$ the only solution of this problem? Explain.

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PART TWO

Problem 5. Suppose $u(x,t)$ is a smooth solution of

$$\begin{aligned} u_t - \Delta u = 0 &\quad \text{in } \Omega \times (0,\infty) \\ u = 0 &\quad \text{on } \partial\Omega \times (0,\infty) \\ u = g &\quad \text{on } \Omega \times \{0\}. \end{aligned}$$

Show that

$$\left\{ \int_\Omega |u(x,t)|^2 dx \right\}^{1/2} \leq e^{-\lambda t} \|g\|_{L^2(\Omega)}$$

for $t \geq 0$, where

$$0 < \lambda := \min_{0 \neq v \in H_0^1(\Omega)} \frac{\|Dv\|^2_{L^2(\Omega)}}{\|v\|^2_{L^2(\Omega)}}.$$

Problem 6. Let $Lu = \Delta u + \sum_{i=1}^n b_i(x) D_i u + c(x)u$. Prove that, if $c < 0$ is bounded in Ω and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies $Lu = f$ in Ω, then

$$\sup_\Omega |u| \leq \sup_{\partial\Omega} |u| + \sup_\Omega |\frac{f}{c}|.$$

Problem 7. Let

$$L = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial}{\partial x_j} \right)$$

be a uniform elliptic operator on Ω with $a_{ij} \in L^\infty(\Omega)$, $a_{ij} = a_{ji}$. Let $V \in L^{n/2}(\Omega)$, $n \geq 3$. Show that, there exists a constant $\varepsilon > 0$ depending only on n, Ω and the ellipticity constant of L such that, if $\|V\|_{L^{n/2}(\Omega)} \leq \varepsilon$, then the bilinear form B associated with $L + V$ satisfies

$$B[u,u] \geq c \|u\|_{H_0^1(\Omega)}$$

for any $u \in H_0^1(\Omega)$, where $c > 0$ depends only on n, Ω and the ellipticity constant of L.

Problem 8. Let

$$L = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial}{\partial x_j} \right)$$

be a uniform elliptic operator on Ω with $a_{ij} \in L^\infty(\Omega)$, $a_{ij} = a_{ji}$. Suppose $u \in H^1(\Omega)$ is a weak solution of $Lu = 0$ in Ω. Show that

$$\int_{B(x_0,r)} |\nabla u(x)|^2 \, dx \leq \frac{C}{(R-r)^2} \int_{B(x_0,R)} |u(x)|^2 \, dx$$

for any $B(x_0,R) \subset \Omega$, where $0 < r < R$ and C depends only on n, $\|a_{ij}\|_\infty$, and the ellipticity constant of L. This is the Caccippoli inequality.